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Modular bootstrap for boundary $\mathcal{N}=2$ Liouville theory

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Abstract: We study the boundary $\mathcal{N}=2$ Liouville theory based on the "modular bootstrap" approach. As fundamental conformal blocks we introduce the "extended characters" that are defined as the proper sums over spectral flows of irreducible characters of the $\mathcal{N}=2$ superconformal algebra (SCA) and clarify their modular transformation properties in models with rational central charges. We then try to classify the Cardy states describing consistent D-branes based on the modular data. We construct the analogues of ZZ-branes [4], localized at the strong coupling region, and the FZZT-branes [2, 3], which extend along the Liouville direction. The former is shown to play important roles to describe the BPS D-branes wrapped around vanishing cycles in deformed Calabi-Yau singularities, reproducing the correct intersection numbers of vanishing cycles. We also discuss the non-BPS D-branes in 2d type-0 (and type-II) string vacua composed of the $\mathcal{N}=2$ Liouville with $\hat{c}(\equiv c/3)=5$. Unstable D0-branes are found as the ZZ-brane analogues mentioned above, and the FZZT-brane analogues are stable due to the existence of mass gap despite the lack of GSO projection.

Keywords: Superstrings and Heterotic Strings, D-branes, Superstring Vacua, Conformal Field Models in String Theory.

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Contents

1. Introduction

Study of D-branes in the $\mathcal{N} = 2$ super-Liouville theory [1] is one of the subjects of great importance by several reasons. First of all, irrational boundary (super)conformal field theories still have many open problems and subtleties, and the $\mathcal{N} = 2$ boundary Liouville theory will yield a good chance to grasp the essence of the issues. Even though the boundary (bosonic) Liouville theory has been firmly understood [2, 3, 4] and the $\mathcal{N} = 1$ boundary Liouville can be investigated in a parallel way [5, 6], the $\mathcal{N} = 2$ Liouville is still hard to analyse and challenging, since it has an extra U(1)-sector coupled with the Liouville field.

The second reason lies in the $\mathcal{N} = 2$ SCFT descriptions of singular Calabi-Yau compactifications or the suitable NS5 configurations in the T-dual picture [7]–[17]. Even under the $g_s \to 0$ limit, the divergence of dilaton at the singular locus makes the relevant conformal system non-trivial. The $\mathcal{N} = 2$ Liouville sector plays a central role in these superconformal system and turning on the Liouville potential deforms the singularity. The boundary state analysis in the $\mathcal{N} = 2$ Liouville theory is thus important to describe the BPS D-branes wrapped around the vanishing cycles. Earlier attempts at this subject have been given in e.g. [18, 19, 20].

The third one originates from recent studies on the proposed dualities between $d \leq 2$ string vacua with matrix models from the viewpoints of unstable D-branes [21]–[41]. The $\mathcal{N} = 2$ Liouville shares the same field content as the 2d type-0 string vacua constructed from the $c = 3/2$ superconformal matter coupled with $\mathcal{N} = 1$ Liouville studied in [26, 27, 32, 35, 39, 41]. However, they have different world-sheet interactions, and hence it is an interesting problem to investigate whether we can discover the similar dual description by matrix models for the $\mathcal{N} = 2$ Liouville. Several analyses and conjectures are found in the recent works [31, 33].

The standard method to determine the proper boundary states in the Liouville theory is to solve the bootstrap equation for the disc one-point function [2, 4]. As we mentioned above, however, this prescription is likely to be difficult for the $\mathcal{N} = 2$ Liouville case. We shall hence take a different route that is easier from a technical point of view: we use the "modular bootstrap" based on the modular transformation properties of character formulas. Although we cannot fix the phase factors of boundary wave functions by this method, we can gain all the necessary informations of cylinder (annulus) amplitudes and classify the Cardy states.

The most non-trivial point in the modular bootstrap for the boundary $\mathcal{N} = 2$ Liouville theory is the following fact: modular transformations of irreducible characters of the $\mathcal{N} = 2$ SCA with $\hat{c}(\equiv c/3) > 1$ generically include the continuous spectrum of U(1)-charges. This feature is not likely to fit to any superstring vacuum on which the space-time supercharges act locally. Our key idea to overcome this difficulty is to consider the *sum over spectral flows* of the irreducible characters as the fundamental ingredients for the bootstrap.

This paper is organized as follows: In section 2 we introduce the "extended characters" that are defined as suitable sums under spectral flow of irreducible characters and examine their modular transformation properties. They are the basic conformal blocks in our modular bootstrap analysis. In section 3 we solve the modular bootstrap equations and construct three types of candidate Cardy states, the "class 1,2 and 3 states". The class 1 branes are found to be natural generalizations of the ZZ-branes [4] in the bosonic Liouville theory and localized at the strong coupling region $\phi \sim +\infty$. The class 2 and 3 branes are regarded as the analogues of the FZZT-branes [2, 3], which extend along the Liouville direction. In section 4 we apply our result to the superconformal systems describing the Calabi-Yau singularities mentioned above. The class 1 branes are identified with (the Liouville sector of) BPS D-branes wrapped around vanishing cycles, and we show that the open string Witten indices reproduce the correct intersection numbers among them. In section 5 we briefly discuss the non-BPS D-branes in the 2d type-0 (and type-II) string vacua composed only of the $\mathcal{N}=2$ Liouville with $\hat{c}=5$. In section 6 we summarize the results of our analysis and make comments on some open issues.

2. Modular data for the $\mathcal{N}=2$ Liouville theory

The $\mathcal{N}=2$ Liouville theory [1] is an $\mathcal{N}=2$ $SCFT_2$ defined as the system of one chiral superfield Φ (one complex boson and one complex fermion) with the linear dilaton of the background charge $\mathcal{Q}$ ($\hat{c}\equiv c/3 = 1+\mathcal{Q}^2$ in our normalization). The real part of bosonic component of Φ is identified as the Liouville field ϕ and the imaginary part Y is associated to the U(1)-current of $\mathcal{N}=2$ SCA. We have two types of the Liouville potentials preserving $\mathcal{N}=2$ superconfomal symmetry; the chiral one S_+ which has the Liouville (imaginary) momentum $1/\mathcal{Q}$ (we also have its anti-chiral counterpart S_-), and the non-chiral one $S_{\rm nc}$ which has the Liouville momentum $\mathcal{Q}$;

$$S_\pm = \int d^2z d^2\theta^\pm \, e^{\frac{1}{\mathcal{Q}}\Phi^\pm} \,, \tag{2.1}$$

$$S_{\rm nc} = \int d^2z d^2\theta^+ d^2\theta^- \, e^{\frac{\mathcal{Q}}{2}(\Phi^+ + \Phi^-)} \,. \tag{2.2}$$

Since our interests are concentrated on cylinder amplitudes, the relevant conformal blocks should be expanded by the irreducible characters of $\mathcal{N}=2$ SCA. Hence we first summarize the character formulas of unitary irreducible representations of $\mathcal{N}=2$ SCA with $\hat{c}>1$ [42, 43]. We here only consider the NS sectors, but one can immediately derive the formulas for other spin structures by the half-integral spectral flows. We denote the conformal weight and U(1)-charge of the highest weight state as h, Q and set $q\equiv e^{2\pi i\tau}$, $y\equiv e^{2\pi i z}$;[1]

(1) massive representations :

$$\mathrm{ch}^{(\mathrm{NS})}(h,Q;\tau,z) = q^{h-(\hat{c}-1)/8}\, y^Q \, \frac{\theta_3(\tau,z)}{\eta(\tau)^3} \,, \qquad \left(h > \frac{|Q|}{2}, \quad 0\le |Q| < \hat{c}-1\right). \tag{2.3}$$

(2) massless matter representations :

$$\mathrm{ch}^{(\mathrm{NS})}_{\mathbf{M}}(Q;\tau,z) = q^{\frac{|Q|}{2}-(\hat{c}-1)/8} y^Q \, \frac{1}{1+y^{\mathrm{sgn}(Q)}q^{1/2}} \, \frac{\theta_3(\tau,z)}{\eta(\tau)^3} \,. \tag{2.4}$$

They correspond to the (anti-)chiral primary state with $h=|Q|/2, \quad (0<|Q|<\hat{c})$.

(3) graviton representation :

$$\mathrm{ch}^{(\mathrm{NS})}_{\mathbf{G}}(\tau,z) = q^{-(\hat{c}-1)/8} \, \frac{1-q}{(1+yq^{1/2})(1+y^{-1}q^{1/2})} \, \frac{\theta_3(\tau,z)}{\eta(\tau)^3} \,. \tag{2.5}$$

They correspond to the vacuum $h=Q=0$, which is the unique state being both chiral and anti-chiral primary.

[1] We borrow the names of representations of $\mathcal{N}=2$ SCA from [44, 45] where string compactifications on Calabi-Yau manifolds have been studied. In the special case $\hat{c}=5$ the string vacuum consists only of the $\mathcal{N}=2$ Liouville sector (together with the superconformal ghost sector), and the "graviton representation" actually corresponds to a tachyon.

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More general unitary representations are generated by the integral spectral flows. Generally the spectral flow generators U_η are defined with a real parameter η by

$$
\begin{aligned}
U_\eta^{-1} L_m U_\eta &= L_m + \eta J_m + \frac{\hat{c}}{2}\eta^2 \delta_{m,0} \,, \\
U_\eta^{-1} J_m U_\eta &= J_m + \hat{c}\eta\delta_{m,0} \,, \\
U_\eta^{-1} G_r^{\pm} U_\eta &= G_{r\pm\eta}^{\pm} \,,
\end{aligned} \tag{2.6}
$$

and we focus on the integer flow parameter $\eta = n \in \mathbb{Z}$ for the time being. The corresponding characters are given by

$$
\mathrm{ch}_*^{(\mathrm{NS})}(*, n; \tau, z) \equiv q^{\frac{\hat{c}}{2}n^2} y^{\hat{c}n}\, \mathrm{ch}_*^{(\mathrm{NS})}(*; \tau, z + n\tau) \,, \qquad (n \in \mathbb{Z}) \,, \tag{2.7}
$$

where $\mathrm{ch}_*^{(\mathrm{NS})}(*; \tau, z)$ is the abbreviated notation of (2.3)-(2.5). Note that the flowed massive character $\mathrm{ch}^{(\mathrm{NS})}(h, Q, n; \tau, z)$ has the same form as (2.3) with the shifts:

$$
h \to h + Qn + \frac{\hat{c}-1}{2}n^2 \,, \qquad Q \to Q + (\hat{c}-1)n \,, \tag{2.8}
$$

(note that $\theta_3(\tau, z + n\tau) = q^{-n^2/2} y^{-n} \theta_3(\tau, z)$). However, the massless characters $\mathrm{ch}_{\mathbf{M}}^{(\mathrm{NS})}(Q, n; \tau, z)$, $\mathrm{ch}_{\mathbf{G}}^{(\mathrm{NS})}(n; \tau, z)$ differ from the original ones (2.4), (2.5) in a non-trivial manner and possess different structures of singular vectors.

The main purpose in this section is to clarify the modular transformation properties of the $\mathcal{N} = 2$ Liouville theory. We first point out that the $\mathcal{N} = 2$ Liouville theory is an $\mathcal{N} = 2$ SCFT of $\hat{c} > 1$ *that possesses no conserved quantities other than the conformal weight and* U(1)*-charge.* (We emphasize that the Liouville field Φ is not free because of the Liouville potential (2.1).). This is analogous to the fact that the bosonic Liouville theory is a CFT with $c > 1$ which has no continuous symmetry other than the Virasoro algebra. Therefore the relevant conformal blocks (for the computation of one-loop vacuum amplitudes) are expected to be the irreducible characters themselves.

However, this is not the whole story: the modular transforms of the characters (2.3), (2.4), (2.5) and (2.7) include continuous spectrum of U(1)-charges. This feature is not likely to be compatible with any superstring vacuum in which the space-time supercharges are well-defined as local operators. This difficulty originates from the irrationality of the relevant conformal system, and the prototype of their resolution has been given in [44, 45, 46]: *we should take the sums of irreducible characters over spectral flows and use these "extended" characters as the fundamental conformal blocks.* Extended characters possess integral U(1)-charges, good modular behaviors and are closed among themselves under modular transformations as in rational conformal field theories (but we need continuous spectrum of conformal weights). Although such extended characters are (infinitely) reducible in the sense of the $\mathcal{N} = 2$ SCA, they are irreducible from the point of view of the "extended chiral algebra", which are defined by adding the spectral flow generators. For example, adding the currents $J^{\pm}$ to the $\mathcal{N} = 2$ algebra at $\hat{c} = 2$ gives the $\mathcal{N} = 4$ SCA of level 1 [44]. We will summarize the modular properties of the extended characters in the cases of $\hat{c} = 2, 3, 4, 5$ in appendix C, which are the special examples of our analysis on the $\mathcal{N} = 2$ Liouville theory given below.

Our main interests in this paper are concentrated on the $\mathcal{N}=2$ Liouville theory in the cases of rational central charges $\hat{c}$. For our later convenience we parameterize the background charge $\mathcal{Q}$ as

$$\mathcal{Q}=\sqrt{\frac{2K}{N}}\,,\qquad \hat{c}=1+\frac{2K}{N}\,, \tag{2.9}$$

where $K\in\mathbb{Z}_{>0}$, $N\in\mathbb{Z}_{>0}$. Different values of N,K give different theories even when $\hat{c}$ is the same, since it is found that they have different spectra of U(1)-charge as we will clarify later.

The models with irrational central charges are not likely to be relevant for the construction of superstring vacua, and are beyond the scope of our study. However, one may also treat the irrational cases by taking the limit $N,K\to\infty$ with tuning the ratio K/N properly, since the following analysis is clearly applicable for arbitrary positive integers N, K. We will derive the modular transformation formulas relevant for this limit at the last of this section.

As stated above, we consider sums over spectral flows of irreducible characters. However, there is still a subtlety due to the fractionality of central charge $\hat{c}$. In fact, the full summation over integral spectral flows yields the theta functions at fractional level K/N (see (C.1) in appendix C), which do not behave well under modular transformations. We thus take the "mod N" *partial* sums over spectral flows, which amounts to adding the spectral flow generators $U_{\pm N}$ (and its superpartners) to the chiral algebra. Essentially the same prescription has been used in [13] in constructing the conformal blocks of toroidal partition function in singular Calabi-Yau manifolds.

Thus we introduce the following extended characters which will be the basic ingredients for our modular bootstrap approach;

$$\chi^{(\mathrm{NS})}(h_0,r,j_0;\tau,z)\equiv\sum_{n\in r+N\mathbb{Z}}q^{\frac{\hat{c}}{2}n^2}y^{\hat{c}n}\,\mathrm{ch}^{(\mathrm{NS})}\left(h_0,Q=\frac{j_0}{N};\tau,z+n\tau\right), \tag{2.10}$$

$$\chi^{(\mathrm{NS})}_{\mathbf{M}}(r,s;\tau,z)\equiv\sum_{n\in r+N\mathbb{Z}}q^{\frac{\hat{c}}{2}n^2}y^{\hat{c}n}\,\mathrm{ch}^{(\mathrm{NS})}_{\mathbf{M}}\left(Q=\frac{s}{N};\tau,z+n\tau\right), \tag{2.11}$$

$$\chi^{(\mathrm{NS})}_{\mathbf{G}}(r;\tau,z)\equiv\sum_{n\in r+N\mathbb{Z}}q^{\frac{\hat{c}}{2}n^2}y^{\hat{c}n}\,\mathrm{ch}^{(\mathrm{NS})}_{\mathbf{G}}\left(\tau,z+n\tau\right), \tag{2.12}$$

where the ranges of parameters r, j_0, s are defined as

$$r\in\mathbb{Z}_N\,,\qquad 0\le j_0\le 2K-1\,,\qquad 1\le s\le N+2K-1\,, \tag{2.13}$$

We shall assume j_0, s are both integers to achieve the locality of spectral flow generators $U_{\pm N}$. We later discuss the construction of superstring vacua by tensoring with rational conformal theories (especially, the $\mathcal{N}=2$ minimal models) and this assumption is necessary for the GSO projection to ensure the integral total U(1)-charges.

In more physical viewpoints one may note that the spectral flow operators $U_{\pm N}$ generate particular winding modes along the U(1)-direction and the assumption of their locality restricts the U(1)-charges as in usual Kaluza-Klein modes. Different choices of the integer parameters N, K yield different spectra of U(1)-charges and correspond to different radii of the compact boson Y, even if $\hat{c}$ is equal, as we already mentioned. We will return this point in section 5 for a special example $\hat{c} = 5$.

Let us present more explicit calculations;

(1) extended massive characters. For the massive characters (2.10) it is convenient to parameterize the characters by the conformal weight h of the flowed vacuum at $n = r$, that is,

$$h \equiv h_0 + \frac{r j_0}{N} + \frac{K r^2}{N} , \tag{2.14}$$

and the summation (2.10) boils down to

$$q^{h - \frac{(j_0 + 2Kr)^2 + K^2}{4NK}} \Theta_{j_0 + 2Kr, NK} \left(\tau, \frac{2z}{N}\right) \frac{\theta_3(\tau, z)}{\eta(\tau)^3} . \tag{2.15}$$

Note that the parameters r, j_0 appear only through the combination $j \equiv j_0 + 2Kr$. It is thus more convenient to define the extended massive characters by

$$\chi^{(\mathrm{NS})}(h, j; \tau, z) = q^{h - \frac{j^2 + K^2}{4NK}} \Theta_{j, NK} \left(\tau, \frac{2z}{N}\right) \frac{\theta_3(\tau, z)}{\eta(\tau)^3} . \tag{2.16}$$

The parameter j runs over the range $j \in \mathbb{Z}_{2NK}$ and the U(1)-charge of vacuum state is equal to $Q = j/N$. By taking the spectral flows of the unitarity condition for $r = 0$ given in (2.3), *i.e.* $h_0 \geq j_0/(2N)$, one can readily find the unitarity condition for general cases [42]

$$h - \frac{j^2 + K^2}{4NK} + \frac{(j_0 - K)^2}{4NK} \geq 0 , \tag{2.17}$$

where j_0 is uniquely determined by the conditions

$$0 \leq j_0 \leq 2K - 1 , \qquad j_0 \equiv j \pmod{2K} . \tag{2.18}$$

(2) extended massless characters. For the massless matter representations (2.4) the spectral flow sum (2.11) becomes

$$\begin{aligned} \chi_{\mathbf{M}}^{(\mathrm{NS})}(r, s; \tau, z) &= \sum_{n \in r + N\mathbb{Z}} q^{\frac{\hat{c}}{2} n^2} y^{\hat{c} n} \, \mathrm{ch}_{\mathbf{M}}^{(\mathrm{NS})} \left(Q = \frac{s}{N}; \tau, z + n\tau\right) \\ &\equiv \sum_{m \in \mathbb{Z}} \frac{\left(y q^{N\left(m + \frac{2r+1}{2N}\right)}\right)^{\frac{s-K}{N}}}{1 + y q^{N\left(m + \frac{2r+1}{2N}\right)}} y^{2K\left(m + \frac{2r+1}{2N}\right)} q^{NK\left(m + \frac{2r+1}{2N}\right)^2} \frac{\theta_3(\tau, z)}{\eta(\tau)^3} . \end{aligned} \tag{2.19}$$

The corresponding vacuum state has the quantum numbers

$$\begin{aligned} h &= \frac{K r^2 + \left(r + \frac{1}{2}\right) s}{N} , \qquad Q = \frac{s + 2Kr}{N} , \qquad \text{for } 0 \leq r < \frac{N}{2} , \\ h &= \frac{K r^2 - \left(r + \frac{1}{2}\right)(N - s)}{N} , \qquad Q = \frac{s - N + 2Kr}{N} , \qquad \text{for } -\frac{N}{2} \leq r \leq -1 . \end{aligned} \tag{2.20}$$

Note that we have $h = Q/2 = s/2N$ when $r = 0$ (chiral primary) and also $h = -Q/2 = \frac{N+2K-s}{2N}$ (anti-chiral primary) when $r = -1$. We also remark "charge conjugation" relation

$$\chi_{\mathbf{M}}^{(\mathrm{NS})}(r, s; \tau, -z) = \chi_{\mathbf{M}}^{(\mathrm{NS})}(-r-1, N+2K-s; \tau, z)\,. \tag{2.21}$$

(3) extended graviton characters. The remaining extended massless characters (2.12) is calculated as

$$\begin{aligned}\chi_{\mathbf{G}}^{(\mathrm{NS})}(r; \tau, z) &= \sum_{n \in r+N\mathbb{Z}} q^{\frac{\hat{c}}{2}n^2} y^{\hat{c}n}\, \mathrm{ch}_{\mathbf{G}}^{(\mathrm{NS})}(\tau, z+n\tau) \\ &\equiv q^{-\frac{K}{4N}} \sum_{m \in \mathbb{Z}} q^{NK\left(m+\frac{r}{N}\right)^2 + N\left(m+\frac{2r-1}{2N}\right)} y^{2K\left(m+\frac{r}{N}\right)+1} \times \\ &\quad \times \frac{1-q}{\left(1+yq^{N\left(m+\frac{2r+1}{2N}\right)}\right)\left(1+yq^{N\left(m+\frac{2r-1}{2N}\right)}\right)} \frac{\theta_3(\tau, z)}{\eta(\tau)^3}\,. \end{aligned} \tag{2.22}$$

We can show the following character identity

$$\begin{aligned}\chi^{(\mathrm{NS})}(h=0, j=0; \tau, z) &= \chi_{\mathbf{G}}^{(\mathrm{NS})}(r=0; \tau, z) + \chi_{\mathbf{M}}^{(\mathrm{NS})}(r=0, s=N; \tau, z) + \\ &\quad + \chi_{\mathbf{M}}^{(\mathrm{NS})}(r=-1, s=2K; \tau, z)\,. \end{aligned} \tag{2.23}$$

Similar character identities have been already known in [44, 46, 47] in the special examples $\hat{c} = 2, 3, 4$. More generally, we obtain by taking the spectral flow of (2.23)

$$\begin{aligned}\chi^{(\mathrm{NS})}\left(h = \frac{Kr^2}{N}, j = 2Kr; \tau, z\right) &= \chi_{\mathbf{G}}^{(\mathrm{NS})}(r; \tau, z) + \chi_{\mathbf{M}}^{(\mathrm{NS})}(r, s=N; \tau, z) + \\ &\quad + \chi_{\mathbf{M}}^{(\mathrm{NS})}(r-1, s=2K; \tau, z)\,, \end{aligned} \tag{2.24}$$

which is quite useful and will be often used in the later analysis.

Now, we are in a position to present the modular transformation formulas which are crucial to our modular bootstrap program. We shall only consider the S-transformations of the NS characters, and use the following abbreviated notations

$$\begin{aligned}\chi(p, j; \tau, z) &\equiv \chi^{(\mathrm{NS})}\left(h = \frac{p^2}{2} + \frac{j^2+K^2}{4NK}, j; \tau, z\right) \\ &\equiv q^{p^2/2}\Theta_{j,NK}\left(\tau, \frac{2z}{N}\right) \frac{\theta_3(\tau, z)}{\eta(\tau)^3}\,, \end{aligned} \tag{2.25}$$

$$\chi_{\mathbf{M}}(r, s; \tau, z) \equiv \chi_{\mathbf{M}}^{(\mathrm{NS})}(r, s; \tau, z)\,, \qquad \chi_{\mathbf{G}}(r; \tau, z) \equiv \chi_{\mathbf{G}}^{(\mathrm{NS})}(r; \tau, z)\,. \tag{2.26}$$

Note that the real "momentum" p here means that no tachyons appear in the spectrum (as is obvious from the last expression in (2.25)). On the other hand, we note that the unitarity condition (2.17) allows a range of pure imaginary values of p;

$$\frac{p^2}{2} \geq -\frac{(j_0-K)^2}{4NK}\,, \qquad j_0 \equiv j\,, \quad (\mathrm{mod}\ 2K)\,, \qquad 0 \leq j_0 \leq 2K-1\,. \tag{2.27}$$

(1) extended massive characters. The S-transformation of the extended massive characters (2.16) is easy. It is reduced to the Fourier transformation of gaussian integral and the familiar modular properties of theta function with the level NK;

$$\chi\left(p, j; -\frac{1}{\tau}, \frac{z}{\tau}\right) = e^{i\pi\frac{\hat{c}z^2}{\tau}} \sqrt{\frac{2}{NK}} \sum_{j' \in \mathbb{Z}_{2NK}} e^{-2\pi i \frac{jj'}{2NK}} \int_0^\infty dp' \cos(2\pi pp')\, \chi\left(p', j'; \tau, z\right). \tag{2.28}$$

(2) extended massless characters. The S-transformation of the extended massless character (2.19) is quite non-trivial.

> *Under the S-transformation the massless representations become the sum of massive and massless representations [44, 46, 48].*

Fortunately a useful formula for the relevant calculations has been provided in [48]. We exhibit it in appendix B. We first note the relation;

$$\chi_{\mathbf{M}}(r, s; \tau, z) = \frac{1}{N} \sum_{j=0}^{N-1} e^{-2\pi i \frac{(2r+1)j}{2N}} I\left(2K/N, j/N, (s-K)/N; \tau, z\right) \frac{\theta_3(\tau, z)}{\eta(\tau)^3}, \tag{2.29}$$

where $I(k, a, b; \tau, z)$ is defined in (B.1). Using the formula (B.2), we can derive the desired modular transformation formula;[2]

$$\begin{aligned}\chi_{\mathbf{M}}\left(r, s; -\frac{1}{\tau}, \frac{z}{\tau}\right) = e^{i\pi\frac{\hat{c}z^2}{\tau}} \Bigg[&\frac{1}{\sqrt{2NK}} \sum_{j' \in \mathbb{Z}_{2NK}} e^{-2\pi i \frac{(s+2Kr)j'}{2NK}} \int_0^\infty dp' \times \\ &\times \frac{\cosh\left(2\pi \frac{N+K-s}{N}\frac{p'}{Q}\right) + e^{i\frac{\pi j'}{K}} \cosh\left(2\pi \frac{s-K}{N}\frac{p'}{Q}\right)}{2\left|\cosh \pi\left(\frac{p'}{Q} + i\frac{j'}{2K}\right)\right|^2} \times \\ &\times \chi\left(p', j'; \tau, z\right) + \frac{i}{N} \sum_{r' \in \mathbb{Z}_N} \sum_{s'=K+1}^{N+K-1} \times \\ &\times e^{-2\pi i \frac{(s+2Kr)(s'+2Kr') - (s-K)(s'-K)}{2NK}} \chi_{\mathbf{M}}(r', s'; \tau, z) + \\ &+ \frac{i}{2N} \sum_{r' \in \mathbb{Z}_N} e^{-2\pi i \frac{(s+2Kr)(2r'+1)}{2N}} \times \\ &\times \left\{\chi_{\mathbf{M}}(r', K; \tau, z) - \chi_{\mathbf{M}}(r', N+K; \tau, z)\right\} \Bigg]. \end{aligned} \tag{2.30}$$

[2] Since the massless characters can be reexpressed as alternating infinite sums of the massive characters, one may wonder how the modular transforms (2.30), (2.31) contain the massless pieces. However, naive infinite sums of the modular coefficients for the massive representations lead to divergences. Mathematically well-defined treatments for the relevant calculus are found in [48], and the "contour deformation technique" used there yields extra pole contributions which generate the massless characters. We also note that the last line in (2.30), which may appear somewhat peculiar, originats from poles on the real axis of p' (and the p'-integral in the first line should be interpreted as the principal value). Similar arguments are also found in [3] for the calculations of cylinder amplitudes.

(3) extended graviton characters. For the graviton representations (2.22), we also obtain

$$
\begin{aligned}
\chi_{\mathbf{G}}\left(r;-\frac{1}{\tau},\frac{z}{\tau}\right) = e^{i\pi\frac{\hat{c}z^2}{\tau}}\Bigg[& \frac{1}{\sqrt{2NK}}\sum_{j'\in\mathbb{Z}_{2NK}} e^{-2\pi i\frac{rj'}{N}} \\
& \times\int_0^\infty dp'\,\frac{\sinh(\pi\mathcal{Q}p')\sinh\left(2\pi\frac{p'}{\mathcal{Q}}\right)}{\left|\cosh\pi\left(\frac{p'}{\mathcal{Q}}+i\frac{j'}{2K}\right)\right|^2}\,\chi\left(p',j';\tau,z\right) \\
& +\frac{2}{N}\sum_{r'\in\mathbb{Z}_N}\sum_{s'=K+1}^{N+K-1}\sin\left(\frac{\pi(s'-K)}{N}\right)\times \\
& \times e^{-2\pi i\frac{r(s'+2Kr')}{N}}\chi_{\mathbf{M}}(r',s';\tau,z)\Bigg]\,,
\end{aligned}
\tag{2.31}
$$

with the help of the character relation (2.24). It is important to note that the parameter s' appearing in the r.h.s.'s of (2.30) and (2.31) runs only over the *partial* range $K \leq s' \leq N+K$, instead of the full range of the unitary (matter) representations $1 \leq s' \leq N+2K-1$. Moreover, graviton representations (2.22) do not appear at all in the r.h.s.'s of (2.30) and (2.31). These features are in a sharp contrast with rational conformal field theories, and have been already found in the $\hat{c}=2$ case [44] (see appendix C). We also point out that all the massless matter representations in this range $K \leq s' \leq N+K$ only include states above the mass gap $\mathcal{Q}^2/8 \equiv K/(4N)$, as is easily checked from the formulas (2.20).

To close this section let us consider the limit $N\to\infty$, $K\to\infty$ with keeping the value $\mathcal{Q}^2 = 2K/N$ fixed. Under this limit the sums over spectral flows (2.10), (2.11), (2.12) should be replaced with the original irreducible characters (2.7) with the identifications $r=n(\in\mathbb{Z})$, $j/N=\omega$ $(-\infty<\omega<+\infty)$, $s/N=\lambda$ $(0<\lambda<\hat{c}(\equiv 1+\mathcal{Q}^2))$;

$$
\begin{aligned}
& \chi^{(\mathrm{NS})}(h,j;\tau,z) \longrightarrow \mathrm{ch}^{(\mathrm{NS})}(h,\omega;\tau,z)(\equiv \mathrm{ch}^{(\mathrm{NS})}(h_0,\omega_0,n;\tau,z)) \\
& \left(\text{where } \omega=\omega_0+\mathcal{Q}^2 n\,,\qquad 0\leq\omega_0<\mathcal{Q}^2\,,\qquad h=h_0+\omega_0 n+\frac{\mathcal{Q}^2}{2}n^2\right) \\
& \chi_{\mathbf{M}}^{(\mathrm{NS})}(r,s;\tau,z) \longrightarrow \mathrm{ch}_{\mathbf{M}}^{(\mathrm{NS})}(\lambda,n;\tau,z)\,,\qquad \chi_{\mathbf{G}}^{(\mathrm{NS})}(r;\tau,z) \longrightarrow \mathrm{ch}_{\mathbf{G}}^{(\mathrm{NS})}(n;\tau,z)
\end{aligned}
\tag{2.32}
$$

The modular transformation formulas are reduced to

$$
\begin{aligned}
& \mathrm{ch}^{(\mathrm{NS})}\left(\frac{p^2}{2}+\frac{\omega^2}{2\mathcal{Q}^2}+\frac{\mathcal{Q}^2}{8},\omega;-\frac{1}{\tau},\frac{z}{\tau}\right) = \\
& = e^{i\pi\frac{\hat{c}z^2}{\tau}}\frac{2}{\mathcal{Q}}\int_{-\infty}^{\infty}d\omega'\int_0^\infty dp' e^{-2\pi i\frac{\omega\omega'}{\mathcal{Q}^2}}\cos(2\pi pp')\times \\
& \times\mathrm{ch}^{(\mathrm{NS})}\left(\frac{p'^2}{2}+\frac{\omega'^2}{2\mathcal{Q}^2}+\frac{\mathcal{Q}^2}{8},\omega';\tau,z\right),
\end{aligned}
\tag{2.33}
$$

$$
\mathrm{ch}_{\mathbf{M}}^{(\mathrm{NS})}\left(\lambda,n;-\frac{1}{\tau},\frac{z}{\tau}\right) =
$$

$$= e^{i\pi\frac{\hat{c}z^2}{\tau}}\left[\frac{1}{\mathcal{Q}}\int_{-\infty}^{\infty}d\omega'\int_0^{\infty}dp' e^{-2\pi i\frac{(\lambda+\mathcal{Q}^2 n)\omega'}{\mathcal{Q}^2}}\times\right.$$
$$\times\frac{\cosh\left(2\pi\left(1+\frac{\mathcal{Q}^2}{2}-\lambda\right)\frac{p'}{\mathcal{Q}}\right)+e^{2\pi i\frac{\omega'}{\mathcal{Q}^2}}\cosh\left(2\pi\left(\lambda-\frac{\mathcal{Q}^2}{2}\right)\frac{p'}{\mathcal{Q}}\right)}{2\left|\cosh\pi\left(\frac{p'}{\mathcal{Q}}+i\frac{\omega'}{\mathcal{Q}^2}\right)\right|^2}\times$$
$$\times\,\mathrm{ch}^{(\mathrm{NS})}\left(\frac{p'^2}{2}+\frac{\omega'^2}{2\mathcal{Q}^2}+\frac{\mathcal{Q}^2}{8},\omega';\tau,z\right)+i\sum_{n'\in\mathbb{Z}}\int_{\frac{\mathcal{Q}^2}{2}}^{1+\frac{\mathcal{Q}^2}{2}}d\lambda'\times$$
$$\left.\times\, e^{-2\pi i\frac{(\lambda+\mathcal{Q}^2 n)(\lambda'+\mathcal{Q}^2 n')-\left(\lambda-\frac{\mathcal{Q}^2}{2}\right)\left(\lambda'-\frac{\mathcal{Q}^2}{2}\right)}{\mathcal{Q}^2}}\mathrm{ch}_{\mathbf{M}}^{(\mathrm{NS})}(\lambda',n';\tau,z)\right], \qquad (2.34)$$

$$\mathrm{ch}_{\mathbf{G}}^{(\mathrm{NS})}\left(n;-\frac{1}{\tau},\frac{z}{\tau}\right)=$$
$$= e^{i\pi\frac{\hat{c}z^2}{\tau}}\left[\frac{1}{\mathcal{Q}}\int_{-\infty}^{\infty}d\omega'\int_0^{\infty}dp' e^{-2\pi i n\omega'}\frac{\sinh(\pi\mathcal{Q}p')\sinh\left(2\pi\frac{p'}{\mathcal{Q}}\right)}{\left|\cosh\pi\left(\frac{p'}{\mathcal{Q}}+i\frac{\omega'}{\mathcal{Q}^2}\right)\right|^2}\times\right.$$
$$\times\,\mathrm{ch}^{(\mathrm{NS})}\left(\frac{p'^2}{2}+\frac{\omega'^2}{2\mathcal{Q}^2}+\frac{\mathcal{Q}^2}{8},\omega';\tau,z\right)+$$
$$+2\sum_{n'\in\mathbb{Z}}\int_{\frac{\mathcal{Q}^2}{2}}^{1+\frac{\mathcal{Q}^2}{2}}d\lambda'\sin\left(\pi\left(\lambda'-\frac{\mathcal{Q}^2}{2}\right)\right)\times$$
$$\left.\times\, e^{-2\pi i n\left(\lambda'+\mathcal{Q}^2 n'\right)}\mathrm{ch}_{\mathbf{M}}^{(\mathrm{NS})}(\lambda',n';\tau,z)\right]. \qquad (2.35)$$

These formulas are relevant for the D-branes in the type-0 string vacua, where we need not impose the integrality of U(1)-charge, as we will see in our later discussions.

3. Modular bootstrap approach to $\mathcal{N}=2$ boundary Liouville theory

3.1 Modular bootstrap: two examples

To begin with, we have to clarify what the "modular bootstrap" means. We present two helpful examples for this purpose. In general (super)conformal field theories are not necessarily rational and there exists a continuous spectrum of primary fields: let us label continuous representations by a parameter p ("momentum") and label the discrete representations by a parameter I. We denote the corresponding characters of the chiral algebra as $\chi_p(\tau)$ and $\chi_I(\tau)$, respectively. The Ishibashi states [49] are defined by imposing suitable boundary conditions on the chiral algebra including the Virasoro algebra;

$$(L_n-\tilde{L}_{-n})\,|p\rangle\rangle=0\,,\qquad (L_n-\tilde{L}_{-n})\,|I\rangle\rangle=0\,, \qquad (3.1)$$

and determined uniquely (up to a phase factor) by the orthonormality conditions as follows;

$$\langle\langle p|\,e^{-\pi TH^{(c)}}\,|p'\rangle\rangle=\delta(p-p')\chi_p(iT)\,,\qquad \langle\langle I|\,e^{-\pi TH^{(c)}}\,|I'\rangle\rangle=\delta_{I,I'}\chi_I(iT)\,,$$
$$\langle\langle p|\,e^{-\pi TH^{(c)}}\,|I\rangle\rangle=0\,, \qquad (3.2)$$

where $H^{(c)} \equiv L_0 + \tilde{L}_0 - c/12$ is the closed string hamiltonian and $\tilde{q} \equiv e^{-2\pi T}$ is the closed string modulus for the cylinder amplitude.

Consistent D-branes are described by the Cardy states [50] of the form

$$|B;\xi\rangle = \int dp\, \Psi_\xi(p)\, |p\rangle\rangle + \sum_I C_\xi(I)\, |I\rangle\rangle\,, \tag{3.3}$$

which should satisfy the following condition exhibiting the open-closed string duality

$$\langle B;\xi_1|e^{-\pi T H^{(c)}}|B;\xi_2\rangle = \int dp\, \rho(p|\xi_1,\xi_2)\chi_p(it) + \sum_I N(I|\xi_1,\xi_2)\chi_I(it)\,. \tag{3.4}$$

In this expression $t \equiv 1/T$ means the open string modulus. The "spectral density" $\rho(p|\xi_1,\xi_2)$ is a positive generalized function (or distribution) and $N(I|\xi_1,\xi_2)$ are positive integers. Note that the positivity of $\rho(p|\xi_1,\xi_2)$ and $N(I|\xi_1,\xi_2)$ yields non-trivial constraints in general.

At first glance, solving the Cardy condition (3.4) appears to be a highly non-trivial problem. We have much more constraints than the number of unknowns. The best we can do is to solve this condition introducing a suitable Ansatz. To proceed further we discuss two examples:

1. The simplest example is the $\mathrm{SU}(2)_k$ WZW model. We only have a discrete spectrum corresponding to the integrable representations of the $\mathrm{SU}(2)_k$ current algebra labelled by ℓ ($\ell = 0, 1, \ldots, k$) and the modular data are given as

$$\begin{aligned} \chi_L^{(k)}\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) &= e^{i\pi \frac{k}{k+2}\frac{z^2}{\tau}} \sum_{\ell=0}^{k} S^{(\mathrm{d})}(\ell|L)\chi_\ell^{(k)}(\tau,z)\,,\\ S^{(\mathrm{d})}(\ell|L) &= \sqrt{\frac{2}{k+2}}\sin\left(\frac{\pi(\ell+1)(L+1)}{k+2}\right)\,, \end{aligned} \tag{3.5}$$

   where the $\mathrm{SU}(2)_k$ character $\chi_\ell^{(k)}(\tau,z)$ is defined in (A.6). Ishibashi states $|\ell\rangle\rangle$ are defined by

$$(J_n^a + \tilde{J}_{-n}^a)\,|\ell\rangle\rangle = 0\,, \qquad \langle\langle \ell|\, e^{-\pi T H^{(c)}} e^{i\pi z(J_0^3 - \tilde{J}_0^3)}\, |\ell'\rangle\rangle = \delta_{\ell,\ell'}\chi_\ell(iT,z)\,, \tag{3.6}$$

   and the Cardy states $|B;L\rangle = \sum_\ell C_L(\ell)\,|\ell\rangle\rangle$ are in a one-to-one correspondence with the integrable representations ($0 \leq \ell,\, L \leq k$) [50]. We note that the Cardy states $|B;L\rangle$ are completely characterized up to phase factors by the following equations, which we would like to call as the "modular bootstrap equations",

$$e^{\pi\frac{k}{k+2}\frac{z^2}{T}}\langle B;0|e^{-\pi T H^{(c)}} e^{i\pi z(J_0^3-\tilde{J}_0^3)}|B;L\rangle = \chi_L(it,z')\,, \qquad ({}^\forall L, t \equiv \frac{1}{T}, z' \equiv -itz)\,. \tag{3.7}$$

   One can easily solve the above equations with an Ansatz $C_L(\ell) = g(\ell)S^{(\mathrm{d})}(\ell|L)$, where $g(\ell)$ are unknown coefficients independent of L. In fact, we readily obtain from (3.7),

$$g(\ell) = \frac{1}{\sqrt{S^{(\mathrm{d})}(\ell|0)}}\,, \qquad C_L(\ell) = \frac{S^{(\mathrm{d})}(\ell|L)}{\sqrt{S^{(\mathrm{d})}(\ell|0)}}\,, \tag{3.8}$$

with a possible phase factor. As is well-known, these solutions satisfy the Cardy condition (3.4) thanks to the Verlinde formula. Note that the special Cardy state $|B;0\rangle$ is characterized by the equation

$$e^{\pi\frac{k}{k+2}\frac{z^2}{T}}\langle B;0|e^{-\pi TH^{(c)}}e^{i\pi z(J_0^3-\tilde{J}_0^3)}|B;0\rangle=\chi_0(it,z')\,,\qquad (t\equiv\frac{1}{T},z'\equiv -itz)\,. \quad (3.9)$$

Since the primary field with $\ell=0$ is the identity, this condition implies that the D-brane described by $|B;0\rangle$ only includes the identity as the boundary operator. We obtain

$$C_0(\ell)=\sqrt{S^{(\mathrm{d})}(\ell|0)}\,. \quad (3.10)$$

Similar constructions can be found in general rational conformal field theories, and the modular bootstrap approach reproduces the correct Cardy states.

2. A more non-trivial example is the boundary Liouville theory. It is known that there exist Cardy states corresponding to both the non-degenerate ("FZZT-branes" [2, 3]) and degenerate representations ("ZZ-brane" [4]). However, the Ishibashi states are only given by the non-degenerate representations above the mass gap. The general Cardy states are written in the form

$$|B;\xi\rangle=\int_0^\infty dp\,\Psi_\xi(p)\,|p\rangle\rangle\,, \quad (3.11)$$

where ξ labels representations of Virasoro algebra and $|p\rangle\rangle$ denotes the Virasoro Ishibashi state with the conformal weight $h=p^2+\frac{\mathcal{Q}^2}{4}$ $(p>0)$. (We here denote the (bosonic) Liouville background charge as $\mathcal{Q}\equiv b+1/b$, $c=1+6\mathcal{Q}^2$, and take the convention $\alpha'=1$.)

The modular data for characters are written in the form as

$$\chi_\xi\left(-\frac{1}{\tau}\right)=\int_0^\infty dp'\,S^{(\mathrm{c})}(p'|\xi)\chi_{p'}(\tau)\,, \quad (3.12)$$

where the coefficients $S^{(\mathrm{c})}(p'|\xi)$ are specified below. Note that the r.h.s. of the above formula contains only the continuous series above the mass gap $h\geq\mathcal{Q}^2/4$.

As in the previous example, Cardy states are characterized by the equations

$$\langle B;O|e^{-\pi TH^{(c)}}|B;\xi\rangle\;=\;\chi_\xi(it)\,, \quad (3.13)$$
$$\langle B;O|e^{-\pi TH^{(c)}}|B;O\rangle\;=\;\chi_{h=0}(it)\,, \quad (3.14)$$

In (3.14), $\chi_{h=0}(it)$ denotes the character of the $h=0$ state (identity operator);

$$\chi_{h=0}(\tau)\equiv\chi_{1,1}(\tau)=\frac{1}{\eta(\tau)}\left(q^{-\frac{1}{4}\left(\frac{1}{b}+b\right)^2}-q^{-\frac{1}{4}\left(\frac{1}{b}-b\right)^2}\right)\,. \quad (3.15)$$

The condition (3.14) again means that the D-brane associated to $|B;O\rangle$ only includes the boundary identity operator. It is identified as the "(1,1)-brane" (a special type of ZZ-brane) defined by the wave function [4]

$$\Psi_O(p) = -2^{5/4}\cdot 2\pi i p\hat{\mu}^{ip/b}\frac{1}{\Gamma(1+i2bp)\Gamma\left(1+i\frac{2p}{b}\right)}\,, \tag{3.16}$$

where $\hat{\mu} \equiv \pi\mu\gamma(b^2)$ ($\gamma(x) \equiv \Gamma(x)/\Gamma(1-x)$) is the "renormalized" cosmological constant. It indeed satisfies the equation (3.14). General Cardy states $|B;\xi\rangle$ are determined by the equation (3.13) as

$$\begin{aligned} f(p) &= \frac{1}{\Psi_O(p)^*} \equiv 2^{-5/4}\frac{1}{2\pi i p}\hat{\mu}^{ip/b}\Gamma(1-i2bp)\Gamma\left(1-i\frac{2p}{b}\right)\,, \\ \Psi_\xi(p) &= f(p)S^{(\mathrm{c})}(p|\xi)\,. \end{aligned} \tag{3.17}$$

As for the FZZT-brane we obtain for the modular coefficients $S^{(\mathrm{c})}(p|\xi)$,

$$S^{(\mathrm{c})}(p|s) = 2\sqrt{2}\cos(2\pi sp)\,. \tag{3.18}$$

In other words $|B;s\rangle$ corresponds to the non-degenerate representation with $h = s^2/4+\mathcal{Q}^2/4$. Based on the bootstrap of disc amplitudes and the perturbative analysis of the Liouville theory it is possible to show that the parameter s is related to the boundary cosmological constant μ_B as $\cosh^2(\pi bs) = \frac{\mu_B^2}{\mu}\sin(\pi b^2)$ [2].

We also obtain

$$S^{(\mathrm{c})}(p|m,n) = 4\sqrt{2}\sinh\left(2\pi\frac{mp}{b}\right)\sinh(2\pi bnp)\,, \qquad (m,n\in\mathbb{Z}_{\geq 0})\,, \tag{3.19}$$

for the general ZZ-brane corresponding to the (m,n)-degenerate representation ($h = -\frac{1}{4}(\frac{m}{b}+nb)^2+\frac{Q^2}{4}$). We note that the relation

$$\Psi_{m,n}(p) = \Psi_{s=i\left(\frac{m}{b}+nb\right)}(p) - \Psi_{s=i\left(\frac{m}{b}-nb\right)}(p)\,, \tag{3.20}$$

follows from the character formula

$$\chi_{m,n}(\tau) = \frac{1}{\eta(\tau)}\left(q^{-\frac{1}{4}\left(\frac{m}{b}+nb\right)^2} - q^{-\frac{1}{4}\left(\frac{m}{b}-nb\right)^2}\right) \equiv \chi_{p=\frac{i}{2}\left(\frac{m}{b}+nb\right)}(\tau) - \chi_{p=\frac{i}{2}\left(\frac{m}{b}-nb\right)}(\tau)\,. \tag{3.21}$$

It is found that the Cardy condition (3.4) is in fact satisfied with the positive spectral densities (under a suitable regularization) at least among the FZZT-branes above the mass gap $s\in\mathbb{R}_+$ and the general ZZ-brane. However, the (m,n)-type ZZ-branes ($(m,n)\neq(1,1)$) generally produce non-unitary representations in the open string channel, and would not be regarded as proper boundary states in string theory.

We emphasize that the solutions (3.16), (3.17) can be reproduced except for the phase factors from the simple equations (3.13), (3.14) with a natural Ansatz $\Psi_\xi(p) = f(p)S^{(\mathrm{c})}(p|\xi)$. Consequently, as long as we work with cylinder amplitudes, the modular bootstrap approach provides all the necessary information and we can classify the Cardy states.

3.2 Modular bootstrap in boundary $\mathcal{N}=2$ Liouville theory and classification of Cardy states

Now, let us study the Cardy states in $\mathcal{N}=2$ Liouville theory based on the modular bootstrap. It is well-known that the $\mathcal{N}=2$ superconformal symmetry allows the following two types of boundary conditions [51];

$$\textbf{A-type}: \quad (J_n - \tilde{J}_{-n})|B\rangle = 0\,, \qquad (G_r^{\pm} - i\tilde{G}_{-r}^{\mp})|B\rangle = 0\,, \tag{3.22}$$

$$\textbf{B-type}: \quad (J_n + \tilde{J}_{-n})|B\rangle = 0\,, \qquad (G_r^{\pm} - i\tilde{G}_{-r}^{\pm})|B\rangle = 0\,, \tag{3.23}$$

Both of them are compatible with the $\mathcal{N}=1$ superconformal symmetry

$$(L_n - \tilde{L}_{-n})|B\rangle = 0\,, \qquad (G_r - i\tilde{G}_{-r})|B\rangle = 0\,, \tag{3.24}$$

where $G = G^+ + G^-$ is the $\mathcal{N}=1$ supercurrent. The following analysis is completely parallel for both A and B-type branes and we shall only consider the A-type from now on.

Encouraged by the successes of the previous examples, let us now discuss our modular bootstrap equations for $\mathcal{N}=2$ Liouville theory. As addressed before, we shall take the extended characters (2.16), (2.19) and (2.22) as the fundamental conformal blocks defining the modular bootstrap equations. This choice is quite natural in realizing the BPS D-branes in supersymmetric string vacua, as we already explained. They have a modular transformation (2.28), (2.30) and (2.31), of the generic form

$$\chi_\xi\left(-\frac{1}{\tau},\frac{z}{\tau}\right) = e^{i\pi\frac{\hat{c}z^2}{\tau}}\left[\int_0^\infty dp' \sum_{j'} S^{(\mathrm{c})}(p',j'|\xi)\chi(p',j';\tau,z) + \right.$$
$$\left. + \sum_{r',s'} S^{(\mathrm{d})}(r',s'|\xi)\chi_{\mathbf{M}}(r',s';\tau,z)\right]. \tag{3.25}$$

In contrast to the previous examples, the modular transform includes *both* continuous and discrete representations in general.

We now propose an Ansatz that the (A-type) Ishibashi states are spanned by the massive and the following massless matter representations, which form the maximal family closed under modular transformations;

$$\begin{aligned} &|p,j\rangle\rangle \qquad (p>0,\ j\in\mathbb{Z}_{2NK})\,,\\ &|r,s\rangle\rangle_{\mathbf{M}} \qquad (r\in\mathbf{Z}_N,\ s=K+1,\ldots,K+N-1)\,, \end{aligned} \tag{3.26}$$

which satisfy the orthogonality conditions

$$\begin{aligned} \langle\langle p,j|\, e^{-\pi T H^{(c)}} e^{i\pi z(J_0+\tilde{J}_0)}\, |p',j'\rangle\rangle &= \delta(p-p')\delta^{(2NK)}_{j,j'}\,\chi(p,j;iT,z)\,,\\ {}_{\mathbf{M}}\langle\langle r,s|\, e^{-\pi T H^{(c)}} e^{i\pi z(J_0+\tilde{J}_0)}\, |r',s'\rangle\rangle_{\mathbf{M}} &= \delta^{(N)}_{r,r'}\delta_{s,s'}\,\chi_{\mathbf{M}}(r,s;iT,z)\,,\\ \langle\langle p,j|\, e^{-\pi T H^{(c)}} e^{i\pi z(J_0+\tilde{J}_0)}\, |r,s\rangle\rangle_{\mathbf{M}} &= 0\,, \end{aligned} \tag{3.27}$$

Here the symbol $\delta^{(M)}_{m,m'}$ means the Kronecker delta mod. M.

Note that among the Ishibashi states

1. we do not include any of the graviton representations,
2. we include massless matter representations only in the range $K+1 \leq s \leq K+N-1$.

It is natural that the graviton representations do not appear in the closed string channel since gravity is decoupled in Liouville theory where all physical states in the closed string sector exist above the mass gap. The graviton representations, however, do appear in the open string channel and the corresponding Cardy states describe basic D-branes of $\mathcal{N}=2$ Liouville theory as we shall see below.

Now we postulate the modular bootstrap equation for $\mathcal{N}=2$ Liouville theory[3]

$$e^{\pi\frac{\hat{c}z^2}{T}}\langle B;O|e^{-\pi TH^{(c)}}e^{i\pi z(J_0+\tilde{J}_0)}|B;\xi\rangle = \chi_\xi(it,z')\,, \tag{3.28}$$

$$e^{\pi\frac{\hat{c}z^2}{T}}\langle B;O|e^{-\pi TH^{(c)}}e^{i\pi z(J_0+\tilde{J}_0)}|B;O\rangle = \chi_{\mathbf{G}}(r=0;it,z')\,, \qquad \left(T\equiv\frac{1}{t}, z'\equiv -itz\right) \tag{3.29}$$

where $\chi_{\mathbf{G}}(r=0;it,z')$ is the character for the graviton representation $h=Q=0$. General Cardy states have the form

$$|B;\xi\rangle = \int_0^\infty dp \sum_{j\in\mathbb{Z}_{2NK}} \Psi_\xi(p,j)\,|p,j\rangle\rangle + \sum_{r\in\mathbb{Z}_N}\sum_{s=K+1}^{N+K-1} C_\xi(r,s)\,|r,s\rangle\rangle_{\mathbf{M}}\,. \tag{3.30}$$

Let us try to solve the bootstrap equations (3.28), (3.29), assuming the Ansatz

$$\Psi_\xi(p,j) = f(p,j)S^{(c)}(p,j|\xi)\,, \qquad C_\xi(r,s) = g(r,s)S^{(d)}(r,s|\xi)\,, \tag{3.31}$$

as in the previous examples. First of all, the basic boundary state $|B;O\rangle$ is readily determined from the modular transformation formula (2.31);

$$|B;O\rangle = \int_0^\infty dp' \sum_{j'\in\mathbb{Z}_{2NK}} \Psi_O(p',j')\,|p',j'\rangle\rangle + \sum_{r'\in\mathbb{Z}_N}\sum_{s'=K+1}^{N+K-1} C_O(r',s')\,|r',s'\rangle\rangle_{\mathbf{M}} \tag{3.32}$$

$$\Psi_O(p',j') = \frac{1}{\mathcal{Q}}\left(\frac{2}{NK}\right)^{1/4}\frac{\Gamma\left(\frac{1}{2}+\frac{j'}{2K}+i\frac{p'}{\mathcal{Q}}\right)\Gamma\left(\frac{1}{2}-\frac{j'}{2K}+i\frac{p'}{\mathcal{Q}}\right)}{\Gamma(i\mathcal{Q}p')\Gamma\left(1+i\frac{2p'}{\mathcal{Q}}\right)}\,, \tag{3.33}$$

$$C_O(r',s') = \left(\frac{2}{N}\right)^{1/2}\sqrt{\sin\left(\frac{\pi(s'-K)}{N}\right)}\,, \tag{3.34}$$

Then from the equations (3.25), (3.28) we can determine

$$f(p',j') \equiv \frac{1}{\Psi_O(p',j')^*} = \mathcal{Q}\left(\frac{NK}{2}\right)^{1/4}\frac{\Gamma(-i\mathcal{Q}p')\Gamma\left(1-i\frac{2p'}{\mathcal{Q}}\right)}{\Gamma\left(\frac{1}{2}+\frac{j'}{2K}-i\frac{p'}{\mathcal{Q}}\right)\Gamma\left(\frac{1}{2}-\frac{j'}{2K}-i\frac{p'}{\mathcal{Q}}\right)}\,,$$

$$g(r',s') \equiv \frac{1}{C_O(r',s')} = \left(\frac{N}{2}\right)^{1/2}\frac{1}{\sqrt{\sin\left(\frac{\pi(s'-K)}{N}\right)}}\,. \tag{3.35}$$

[3]If we take the B-type boundary conditions, we instead make an insertion of $e^{i\pi z(J_0-\tilde{J}_0)}$.

In the case of general graviton representations $|B;r\rangle$ one has

$$\Psi_r(p',j') = e^{-2\pi i \frac{rj'}{N}} \Psi_O(p',j'), \ \ C_r(r',s') = e^{-2\pi i \frac{r(s'+2Kr')}{N}} C_O(r',s')\,, \tag{3.36}$$

from (3.31), (3.35) and the modular transformation formula (2.31).

As for the massless matter representations, there is a slight difficulty in deriving the corresponding Cardy states. This is because Ishibashi states are spanned by $|r,s\rangle\rangle_M$ with $1 \leq s \leq N+2K-1$ while there appear the extra "boundary terms" $\chi_{\mathbf{M}}(r, s=K)$ and $\chi_{\mathbf{M}}(r, s=N+K)$ in the r.h.s. of (2.30). We must thus combine the characters so that the terms $\chi_{\mathbf{M}}(r, s=K)$ and $\chi_{\mathbf{M}}(r, s=N+K)$ cancel in their modular transforms. The minimal combinations with such a property is given by a pair of representations with U(1) charges differing by an odd integer

$$\begin{aligned}
&[(r_1,s_1),(r_2,s_2)] \longleftrightarrow \chi_{\mathbf{M}}(r_1,s_1;\tau,z) + \chi_{\mathbf{M}}(r_2,s_2;\tau,z)\,, \\
&(s_1+2Kr_1) - (s_2+2Kr_2) \in N(2\mathbb{Z}+1)\,.
\end{aligned} \tag{3.37}$$

For such a pair of representations (3.37) we can solve (3.28) in the form

$$\begin{aligned}
\Psi_{[(r_1,s_1),\,(r_2,s_2)]}(p,j) \;&=\; f(p,j)\left(S^{(\mathrm{c})}(p,j|r_1,s_1) + S^{(\mathrm{c})}(p,j|r_2,s_2)\right), \\
C_{[(r_1,s_1),\,(r_2,s_2)]}(r,s) \;&=\; g(r,s)\left(S^{(\mathrm{d})}(r,s|r_1,s_1) + S^{(\mathrm{d})}(r,s|r_2,s_2)\right),
\end{aligned} \tag{3.38}$$

where the modular coefficients $S^{(\mathrm{c})}(p,j|r_i,s_i)$, $S^{(\mathrm{d})}(r,s|r_i,s_i)$ are read off from the formula (2.30).

In this way, we obtain three types of candidate Cardy states

- **class 1**: Boundary states associated to the graviton representations $\chi_{\mathbf{G}}(r;\tau,z)$, which we denote as $|B;r\rangle$ ($r \in \mathbb{Z}_N$). Especially, $|B;r=0\rangle \equiv |B;O\rangle$.

- **class 2**: Boundary states associated to the massive representations $\chi(p,j;\tau,z)$, which we denote as $|B;p,j\rangle$. The parameters p, j have to be constrained by the unitarity condition (2.27).

- **class 3**: Boundary states associated to pairs of massless matter representations $\chi_{\mathbf{M}}(r_1,s_1;\tau,z) + \chi_{\mathbf{M}}(r_2,s_2;\tau,z)$ (3.37), which we denote as $|B;(r_1,s_1),(r_2,s_2)\rangle$.

Of course this is not the whole story. We must check that these candidate states in fact satisfy the Cardy condition (3.4). It is a non-trivial task to check whether the overlaps $\langle B;\xi_1(\neq O)|e^{-\pi T H^{(c)}}|B;\xi_2(\neq O)\rangle$ can be rewritten in the form (3.4) with positive spectral densities. We leave the analysis of these cylinder amplitudes in the next subsection, and present several comments here:

1. Since $\Psi_O(p,j=0)$ has a simple zero at $p=0$, the solutions of (3.28) have an ambiguity. We may add the term of the form $\sim c\delta(p)$ to the function $f(p,0)$. This would lead to a subtlety, because if allowing such term, the overlap amplitudes of boundary states would be ill-defined (due to the product of delta functions). In the

bosonic (and $\mathcal{N}=1$) Liouville theory, it is known from the bootstrap analyses for disk one-point functions that no such delta function terms appear in the boundary wave functions [2, 4, 5, 6]. We believe they do not exist in our $\mathcal{N}=2$ case either and shall neglect this possibility from here on.

2. Our modular bootstrap approach determines only the absolute values of the boundary wave functions and there exist phase ambiguities in the solutions (3.33), (3.34). These phases may depend on the values of the cosmological constant. In particular, determination of the phase factor for $\Psi_O(p,j)$ (or equivalently that of $f(p,j)$) is an important problem since it could reproduce the disk amplitudes of general vertex operators by suitable analytic continuations.

   As is well-known, the boundary wave function of a general Cardy state $|B;\xi\rangle$ may be interpreted as the disk one-point function by a relation such as

$$\langle e^{\beta\phi(z,\bar{z})}\rangle_{\mathrm{disc}} \approx \frac{\Psi_\xi^*(p,0)}{|z-\bar{z}|^{2h_\beta}}, \qquad (\beta \equiv \frac{\mathcal{Q}}{2}+ip), \tag{3.39}$$

   where $e^{\beta\phi}$ is the primary field of conformal weight $h_\beta=-\frac{\beta(\beta-\mathcal{Q})}{2}\equiv\frac{p^2}{2}+\frac{\mathcal{Q}^2}{8}$.

   A possible way to determine the phase is to make use of the "reflection relation" [2];

$$\Psi_O(-p,j)=R(p,j)\Psi_O(p,j), \tag{3.40}$$

   where $R(p,j)$ denotes the reflection amplitude (bulk two point function). In [52, 53] a solution for the reflection amplitude has been proposed based on the analysis parallel to [54] for the model with the Liouville potential $\mu(S_+ + S_-)$, $\mu\in\mathbb{R}_{>0}$.[4] It is rewritten in our convention as

$$R(p,j)=\hat{\mu}^{-2i\mathcal{Q}p}\frac{\Gamma(i\mathcal{Q}p)\Gamma\left(1+i\frac{2p}{\mathcal{Q}}\right)\Gamma\left(\frac{1}{2}+\frac{j}{2K}-i\frac{p}{\mathcal{Q}}\right)\Gamma\left(\frac{1}{2}-\frac{j}{2K}-i\frac{p}{\mathcal{Q}}\right)}{\Gamma(-i\mathcal{Q}p)\Gamma\left(1-i\frac{2p}{\mathcal{Q}}\right)\Gamma\left(\frac{1}{2}+\frac{j}{2K}+i\frac{p}{\mathcal{Q}}\right)\Gamma\left(\frac{1}{2}-\frac{j}{2K}+i\frac{p}{\mathcal{Q}}\right)}, \tag{3.41}$$

   where $\hat{\mu}$ is the renormalized cosmological constant proportional to μ whose precise value is not important here.

   If we compare (3.33) with the reflection amplitude (3.41), we find that they are completely consistent with the relation (3.40) provided $\Psi_O(p,j)$ is multiplied by an extra phase factor $\hat{\mu}^{i\mathcal{Q}p}$ as

$$\Psi_O(p,j)=\hat{\mu}^{i\mathcal{Q}p}\frac{1}{\mathcal{Q}}\left(\frac{2}{NK}\right)^{1/4}\frac{\Gamma\left(\frac{1}{2}+\frac{j}{2K}+i\frac{p}{\mathcal{Q}}\right)\Gamma\left(\frac{1}{2}-\frac{j}{2K}+i\frac{p}{\mathcal{Q}}\right)}{\Gamma(i\mathcal{Q}p)\Gamma\left(1+i\frac{2p}{\mathcal{Q}}\right)}. \tag{3.42}$$

   We note that this phase factor is what is expected based on the simple scaling argument for the (fractional) number of insertions of bulk Liouville potential terms for the one-point function (3.39). Thus we suggest that (3.42) is in fact the correct expression of the boundary wave function of $\mathcal{N}=2$ Liouville theory.

[4] We should thank Y. Nakayama for drawing our attention to the paper [53]. We should also thank P. Baseilhac for informing us of [52].

3. As is obvious from our construction, the class 1 boundary states are regarded as generalizations of the (1,1)-type ZZ state in the bosonic Liouville theory [4]. Especially the wave function $\Psi_O(p,j)$ has a simple zero at $p=0$. This fact implies that the corresponding D-brane is localized at the strong coupling region $\phi \sim +\infty$ as discussed in [22]. Furthermore, we find that $\Psi_O^*(p,0)$ has a simple pole at $ip = \mathcal{Q}/2$, corresponding to the one point function of $e^{\mathcal{Q}\phi}$. Since the wave function for the quantum state $e^{\mathcal{Q}\phi}|0\rangle$ behaves as $\psi_{\mathcal{Q}}(\phi) \sim g_s^{-1}e^{\mathcal{Q}\phi} \sim e^{\frac{\mathcal{Q}}{2}\phi}$ [55], the existence of this pole is a signal such that the brane of $|B;O\rangle$ is really located at $\phi \sim +\infty$.

4. The class 2 and 3 states appear to describe analogues of the FZZT branes [2, 3] which are extended along the Liouville direction. We have a simple pole at $p=0$ this time, suggesting the "Neumann" boundary condition along the Liouville direction. Poles of $\Psi_\xi^*(p,j)$ are determined by the function $f(p,j)^*$ which has the following form

$$f(p,j)^* \approx \mu^{-\frac{\mathcal{Q}}{2}ip-\frac{j}{2N}}\bar{\mu}^{-\frac{\mathcal{Q}}{2}ip+\frac{j}{2N}}\frac{\Gamma(i\mathcal{Q}p)\Gamma\left(1+i\frac{2p}{\mathcal{Q}}\right)}{\Gamma\left(\frac{1}{2}+\frac{j}{2K}+i\frac{p}{\mathcal{Q}}\right)\Gamma\left(\frac{1}{2}-\frac{j}{2K}+i\frac{p}{\mathcal{Q}}\right)}\,. \tag{3.43}$$

Here we have also introduced an extra phase factor as in (3.42) (with complex cosmological constants μ and $\bar{\mu}$). (3.43) implies that the one point function (3.39) has simple poles at $ip = -\frac{n}{\mathcal{Q}}$, $(n \in \mathbb{Z}_{\geq 0})$. In the perturbative approach to Liouville theory, these poles could be interpreted as the results of an *integral* number of insertions of the boundary cosmological constant operators $S_\pm^{(B)}$ which are defined like the bulk operators (2.1) but with a half Liouville momenta.[5] In fact, the location of these poles as well as the factor $\mu^{n/2}\bar{\mu}^{n/2} \equiv |\mu|^n$ fit nicely to the amplitude $\langle e^{\beta\phi}\,(S_+^{(B)}S_-^{(B)})^n\rangle_{0,\mathrm{disc}}$ where the subscript "0" indicates the free field calculation. Note that we need the same number of $S_+^{(B)}$ and $S_-^{(B)}$ insertions to obtain a non-vanishing result due to U(1)-charge conservation in free field calculation. These facts support our suggestion that the class 2 and class 3 states describe the analogues of FZZT-branes. Of course, these expectations should be confirmed by the bootstrap calculation for the disc one-point functions, which will clarify the precise interpretations of these boundary states based on the boundary Liouville interactions.

5. Since $|B;O\rangle$ only includes the Ishibashi states in the NS sector, we cannot determine the R-sector Cardy states based on the equation (3.28). We can instead determine them by means of the half-integral spectral flow, as we will do explicitly in the next section. The boundary wave functions for the R-sector determined this way have expected forms including the modular coefficients of $\chi_*^{(\widetilde{\mathrm{NS}})}(-1/\tau, z/\tau)$ rather than $\chi_*^{(\mathrm{NS})}(-1/\tau, z/\tau)$.

3.3 Cylinder amplitudes and Cardy condition

Let us make an analysis on the overlaps (cylinder amplitudes) of the candidate boundary states to examine the Cardy condition. The calculation is straightforward and we will see

[5] Recently, a detailed study on the explicit forms of $S_\pm^{(B)}$ has been given in [56].

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the relation (3.4) is satisfied in all the cases except for the overlaps between the class 3 states, that is, of the type $\langle B;(r_1,s_1),\,(r_2,s_2)|e^{-\pi TH^{(c)}}e^{i\pi z(J_0+\tilde{J}_0)}|B;(r_1',s_1'),\,(r_2',s_2')\rangle$. It is rather non-trivial to check the Cardy condition in this case and we will present its analyses in appendix D. In this section we restrict ourselves to the class 1 and 2 boundary states. We assume that the momentum p labeling the class 2 states is real for the time being, and later discuss the cases of imaginary p (bounded by the unitarity condition (2.27)).

First of all, the class 1 states have the following overlap amplitudes

$$e^{\pi\frac{\hat{c}z^2}{T}}\langle B;r'|e^{-\pi TH^{(c)}}e^{i\pi z(J_0+\tilde{J}_0)}|B;r\rangle = \chi_{\mathbf{G}}(r-r';it,z')\,, \tag{3.44}$$

$$e^{\pi\frac{\hat{c}z^2}{T}}\langle B;r'|e^{-\pi TH^{(c)}}e^{i\pi z(J_0+\tilde{J}_0)}|B;p,j\rangle = \chi(p,j-2Kr';it,z')\,, \tag{3.45}$$

$$\begin{aligned} e^{\pi\frac{\hat{c}z^2}{T}}\langle B;r'|e^{-\pi TH^{(c)}}e^{i\pi z(J_0+\tilde{J}_0)}|B;(r_1,s_1),(r_2,s_2)\rangle &= \chi_{\mathbf{M}}(r_1-r',s_1;it,z')+ \\ &\quad +\chi_{\mathbf{M}}(r_2-r',s_2;it,z')\,, \end{aligned} \tag{3.46}$$

where the open string modulus t, z' are defined in (3.28), (3.29).

We next calculate the overlaps not including the class 1 states

$$\begin{aligned} &e^{\pi\frac{\hat{c}z^2}{T}}\langle B;p_1,j_1|e^{-\pi TH^{(c)}}e^{i\pi z(J_0+\tilde{J}_0)}|B;p_2,j_2\rangle = \\ &\quad = \int_0^\infty dp\Big[\rho_1(p|p_1,p_2)\chi(p,j_2-j_1;it,z')+\rho_2(p|p_1,p_2)\times \\ &\qquad\qquad \times\left\{\chi(p,j_2-j_1+N;it,z')+\chi(p,j_2-j_1-N;it,z')\right\}\Big]\,, \end{aligned} \tag{3.47}$$

$$\begin{aligned} \rho_1(p|p_1,p_2) &= \int_0^\infty dp'\frac{\cos(2\pi pp')}{\sinh(\pi\mathcal{Q}p')\sinh\left(\frac{2\pi p'}{\mathcal{Q}}\right)}\times \\ &\quad\times\sum_{\epsilon_i=\pm1}\cosh\left(2\pi\left(\frac{1}{\mathcal{Q}}+i\epsilon_1p_1+i\epsilon_2p_2\right)p'\right)\,, \end{aligned} \tag{3.48}$$

$$\rho_2(p|p_1,p_2) = \frac{1}{2}\int_0^\infty dp'\,\frac{\cos(2\pi pp')}{\sinh(\pi\mathcal{Q}p')\sinh\left(\frac{2\pi p'}{\mathcal{Q}}\right)}\sum_{\epsilon=\pm1}\cos\left(2\pi\left(p_1+\epsilon p_2\right)p'\right)\,, \tag{3.49}$$

and also

$$\begin{aligned} &e^{\pi\frac{\hat{c}z^2}{T}}\langle B;p_1,j_1|e^{-\pi TH^{(c)}}e^{i\pi z(J_0+\tilde{J}_0)}|B;(r_1,s_1),(r_2,s_2)\rangle = \\ &\quad = \sum_{i=1,2}\int_0^\infty dp\Big[\hat{\rho}_1(p|p_1,s_i)\chi(p,(s_i+2Kr_i)-j_1;it,z') \\ &\qquad\qquad + \hat{\rho}_2(p|p_1,s_i)\chi(p,(s_i-N+2Kr_i)-j_1;it,z')\Big]\,, \end{aligned} \tag{3.50}$$

$$\hat{\rho}_1(p|p_1,s) = \int_0^\infty dp'\frac{\cos(2\pi pp')}{\sinh(\pi\mathcal{Q}p')\sinh(\frac{2\pi p'}{\mathcal{Q}})}\sum_{\epsilon=\pm1}\cosh\left(2\pi\left(\frac{N+K-s}{N}\frac{1}{\mathcal{Q}}+i\epsilon p_1\right)p'\right) \tag{3.51}$$

$$\hat{\rho}_2(p|p_1,s) = \int_0^\infty dp'\frac{\cos(2\pi pp')}{\sinh(\pi\mathcal{Q}p')\sinh(\frac{2\pi p'}{\mathcal{Q}})}\sum_{\epsilon=\pm1}\cosh\left(2\pi\left(\frac{s-K}{N}\frac{1}{\mathcal{Q}}+i\epsilon p_1\right)p'\right)\,. \tag{3.52}$$

The spectral densities (3.48), (3.49), (3.51) and (3.52) have similar forms as in the case of FZZT-branes. The momentum integral has a divergence at $p' = 0$ as in the bosonic Liouville case. Such a divergence is not surprising, since the Liouville direction is non-compact and the class 2 and 3 branes correspond to the Neumann boundary condition along this direction. We note that the coefficients of $1/p'^2$ of the integrand are all positive and it is easy to check the positivity of the spectral functions.

After subtracting off the divergent piece which is independent of the boundary states we find that the convergent part of spectral densities are written in terms of the "q-gamma functions" defined in [2] (see (A.4));

$$\rho_1(p|p_1,p_2) \approx \frac{i}{2\pi}\sum_{\epsilon_i=\pm 1} \epsilon_0 \partial_p \ln S_{\mathcal{Q}/\sqrt{2}}\left(\frac{1}{\sqrt{2}}\left(\frac{\mathcal{Q}}{2} + i\epsilon_0 p + i\epsilon_1 p_1 + i\epsilon_2 p_2\right)\right), \tag{3.53}$$

$$\rho_2(p|p_1,p_2) \approx \frac{i}{2\pi}\sum_{\epsilon_i=\pm 1} \partial_p \ln S_{\mathcal{Q}/\sqrt{2}}\left(\frac{1}{\sqrt{2}}\left(\frac{\mathcal{Q}}{2} + \frac{1}{\mathcal{Q}} + ip + i\epsilon_1 p_1 + i\epsilon_2 p_2\right)\right), \tag{3.54}$$

$$\hat{\rho}_1(p|s,p_1) \approx \frac{i}{2\pi}\sum_{\epsilon_i=\pm 1} \epsilon_0 \partial_p \ln S_{\mathcal{Q}/\sqrt{2}}\left(\frac{1}{\sqrt{2}}\left(\frac{\mathcal{Q}}{2} + \frac{1}{\mathcal{Q}}\frac{s-K}{N} + i\epsilon_0 p + i\epsilon_1 p_1\right)\right), \tag{3.55}$$

$$\hat{\rho}_2(p|s,p_1) \approx \frac{i}{2\pi}\sum_{\epsilon_i=\pm 1} \epsilon_0 \partial_p \ln S_{\mathcal{Q}/\sqrt{2}}\left(\frac{1}{\sqrt{2}}\left(\frac{\mathcal{Q}}{2} + \frac{1}{\mathcal{Q}}\frac{N+K-s}{N} + i\epsilon_0 p + i\epsilon_1 p_1\right)\right). \tag{3.56}$$

The q-gamma function appears in the calculations of disc two-point functions (or the reflection coefficients) in the $\mathcal{N} = 0$ and $\mathcal{N} = 1$ Liouville theories, and the quantum mechanical relation between the two-point functions and the spectral densities has been proposed in [4]. It is an interesting problem to compute the disc two-point functions in the $\mathcal{N} = 2$ Liouville theory and compare them with the spectral densities we have derived here.

We finally discuss the class 2 states with pure imaginary p. Note that we allow only real values of momenta p for Ishibashi states, however, imaginary values of momenta p possibly appear among Cardy states. If one considers such a state, the integral over p' in (3.48), (3.49), (3.51), (3.52) may generate additional divergences at $p' = \infty$. One may eliminate such a divergence by shifting the contour of p integration in (3.47), (3.50). When one shifts back the contour of p to the real axis, one picks up additional contributions from the poles of the q-gamma functions as discussed in [3]. These discrete terms correspond to the non-degenerate representations below the mass gap and appear with positive integer coefficients. (Actually, we find the factor +1 per each pole.) However, they do not always satisfy the unitarity bound (2.27). Although the complete classification of the class 2 states compatible with unitarity is an interesting and tractable problem, we here only point out the following fact: if $|p| < \mathcal{Q}/4$, the class 2 states with imaginary p do not contain the discrete terms in arbitrary overlaps with them. In these cases overlap amplitudes are obviously compatible with unitarity.

In this way, we have established that all the class 1 states and class 2 states satisfying

$$\frac{p^2}{2}\begin{cases} \geq -\dfrac{(j_0-K)^2}{4NK}\,, & \text{if } (j_0-K)^2 < \dfrac{K^2}{4} \\ > -\dfrac{K}{16N}\,, & \text{if } (j_0-K)^2 \geq \dfrac{K^2}{4} \end{cases}, \quad 0 \leq j_0 < 2K\,, \quad j_0 \equiv j \pmod{2K} \tag{3.57}$$

are compatible with the Cardy condition (3.4) and hence describe consistent D-branes in the general $\mathcal{N}=2$ Liouville theory.

3.4 Summary of Cardy states in $\mathcal{N}=2$ Liouville theory

To close this section let us present a summary of the (NS) Cardy states in $\mathcal{N}=2$ Liouville theory we propose for the convenience of readers.

General form:

$$|B;\xi\rangle = \int_0^\infty dp' \sum_{j\in\mathbb{Z}_{2NK}} \Psi_\xi(p',j')\,|p',j'\rangle\rangle + \sum_{r\in\mathbb{Z}_N}\sum_{s'=K+1}^{N+K-1} C_\xi(r',s')\,|r',s'\rangle\rangle_{\mathbf{M}}\,, \tag{3.58}$$

where the Ishibashi states $|p',j'\rangle\rangle$, $|r',s'\rangle\rangle_{\mathbf{M}}$ are defined in (3.26), (3.27). The boundary wave function $\Psi_\xi(p',j')$ could include the scaling factor proportional to $\mu^{-\frac{\mathcal{Q}}{2}ip'-\frac{j'}{2N}}$ $\bar{\mu}^{-\frac{\mathcal{Q}}{2}ip'+\frac{j'}{2N}}$ (irrespective of ξ), which is just a phase factor for real p'.

- **class 1 states**: $|B;r\rangle$ $(r\in\mathbb{Z}_N)$

$$\begin{aligned} \Psi_r(p',j') &= \frac{1}{\mathcal{Q}}\left(\frac{2}{NK}\right)^{1/4} e^{-2\pi i\frac{rj'}{N}} \frac{\Gamma\left(\frac{1}{2}+\frac{j'}{2K}+i\frac{p'}{\mathcal{Q}}\right)\Gamma\left(\frac{1}{2}-\frac{j'}{2K}+i\frac{p'}{\mathcal{Q}}\right)}{\Gamma(i\mathcal{Q}p')\Gamma\left(1+i\frac{2p'}{\mathcal{Q}}\right)}\,, \\ C_r(r',s') &= \left(\frac{2}{N}\right)^{1/2} e^{-2\pi i\frac{r(s'+2Kr')}{N}} \sqrt{\sin\left(\frac{\pi(s'-K)}{N}\right)}\,. \end{aligned} \tag{3.59}$$

- **class 2 states**: $|B;p,j\rangle$

$$\frac{p^2}{2} \geq -\frac{(j_0-K)^2}{4NK}\,, \qquad j_0\equiv j\,, \pmod{2K}\,, \qquad 0\leq j_0 \leq 2K-1\,. \tag{3.60}$$

$$\begin{aligned} \Psi_{p,j}(p',j') &= \mathcal{Q}\left(\frac{2}{NK}\right)^{1/4} e^{-2\pi i\frac{jj'}{2NK}}\cos(2\pi pp') \frac{\Gamma(-i\mathcal{Q}p')\Gamma\left(1-i\frac{2p'}{\mathcal{Q}}\right)}{\Gamma\left(\frac{1}{2}+\frac{j'}{2K}-i\frac{p'}{\mathcal{Q}}\right)\Gamma\left(\frac{1}{2}-\frac{j'}{2K}-i\frac{p'}{\mathcal{Q}}\right)}\,, \\ C_{p,j}(r',s') &= 0\,. \end{aligned} \tag{3.61}$$

- **class 3 states**: $|B;(r_1,s_1),(r_2,s_2)\rangle$

$$r_i\in\mathbb{Z}_N\,, \quad 1\leq s_i\leq N+2K-1\,, \quad (s_1+2Kr_1)-(s_2+2Kr_2)\in N(2\mathbb{Z}+1)\,.$$

$$\Psi_{(r_1,s_1),(r_2,s_2)}(p',j') = \mathcal{Q}\left(\frac{NK}{2}\right)^{1/4}\left(S^{(c)}(p',j'|r_1,s_1)+S^{(c)}(p',j'|r_2,s_2)\right)\times$$

$$\times \frac{\Gamma(-iQp')\Gamma\left(1-i\frac{2p'}{Q}\right)}{\Gamma\left(\frac{1}{2}+\frac{j'}{2K}-i\frac{p'}{Q}\right)\Gamma\left(\frac{1}{2}-\frac{j'}{2K}-i\frac{p'}{Q}\right)},$$

$$C_{(r_1,s_1),(r_2,s_2)}(r',s') = \left(\frac{N}{2}\right)^{1/2} \frac{1}{\sqrt{\sin\left(\frac{\pi(s'-K)}{N}\right)}} \times$$

$$\times \left(S^{(\mathrm{d})}(r',s'|r_1,s_1) + S^{(\mathrm{d})}(r',s'|r_2,s_2)\right),$$

$$S^{(\mathrm{c})}(p',j'|r,s) = \frac{1}{\sqrt{2NK}} e^{-2\pi i \frac{(s+2Kr)j'}{2NK}} \times$$

$$\times \frac{\cosh\left(2\pi\frac{N+K-s}{N}\frac{p'}{Q}\right) + e^{i\frac{\pi j'}{K}}\cosh\left(2\pi\frac{s-K}{N}\frac{p'}{Q}\right)}{2\left|\cosh \pi\left(\frac{p'}{Q}+i\frac{j'}{2K}\right)\right|^2},$$

$$S^{(\mathrm{d})}(r',s'|r,s) = \frac{i}{N} e^{-2\pi i \frac{(s+2Kr)(s'+2Kr')-(s-K)(s'-K)}{2NK}}. \tag{3.62}$$

4. Coupling with $\mathcal{N}=2$ minimal models and Calabi-Yau singularities

As an application of our previous analysis let us consider the superstring vacua of the form;

$$\mathbb{R}^{d-1,1} \times \left[\mathcal{N}=2 \text{ minimal model of level } k \left(\hat{c} = \frac{k}{(k+2)}\right)\right] \times$$
$$\times \left[\mathcal{N}=2 \text{ Liouville } \left(\hat{c} = 1+\tfrac{2K}{N}\right)\right], \tag{4.1}$$

with the criticality condition

$$\frac{d}{2} + \frac{k}{k+2} + \left(1+\frac{2K}{N}\right) = 5. \tag{4.2}$$

Precisely speaking, we have to make an orbifoldization of this theory and project onto sectors with integral total U(1)-charges. This procedure is often called the "GSO projection".[6] These superconformal systems are believed to describe the type-II string theories compactified on the Calabi-Yau n-folds ($n \equiv (10-d)/2$) with isolated A-D-E singularities when one uses modular invariants of the $\mathcal{N}=2$ minimal model of the corresponding A-D-E type [9, 10, 13, 16]. For simplicity we shall work only with the A-type modular invariants, which correspond to the A_{k+1}-type Calabi-Yau singularity locally expressed as $X^{k+2} + z_1^2 + \cdots + z_n^2 = 0$ (for CY_n). The main purpose of this section is to clarify how we can identify the vanishing cycles of these singularities with appropriate Cardy states. Earlier studies on this subject has been given in e.g. [18, 19, 20].

The criticality condition (4.2) leads to the following values of N, K

- $d=6$: $N=k+2$, $K=1$.
- $d=4$:

[6] After projection to integral U(1) charges, we must further impose the usual GSO projection with respect to the spin structures, which leads to, for instance, $Q_{\mathrm{total}} \in 2\mathbb{Z}+1$ for the NS sector.

- $\cdot$ $k \in 2\mathbb{Z}_{\geq 0}$: $N = k+2$, $K = (k+4)/2$,
- $\cdot$ $k \in 2\mathbb{Z}_{\geq 0}+1$: $N = 2(k+2)$, $K = k+4$.

- $d = 2$: $N = k+2$, $K = k+3$.

4.1 BPS D-branes wrapped around vanishing cycles

Now, we start the analysis on the BPS D-branes in these superconformal systems. We again concentrate on the A-type boundary states which describe the special lagrangian cycles..

First of all, the Cardy states in the minimal model are well-known (see, for instance, [57]). Let $|\ell,m\rangle\rangle^{(\mathrm{NS})}$ ($|\ell,m\rangle\rangle^{(\mathrm{R})}$) be the Ishibashi states in the NS (R) sector characterized by

$$\begin{aligned}
{}^{(\mathrm{NS})}\langle\langle \ell,m|\, e^{-\pi T H^{(c)}} e^{i\pi z(J_0+\tilde{J}_0)}\, |\ell',m'\rangle\rangle^{(\mathrm{NS})} &= \left(\delta_{\ell,\ell'}\delta_{m,m'} + \delta_{\ell,k-\ell'}\delta_{m,m'+k+2}\right) \mathrm{ch}^{(\mathrm{NS})}_{\ell,m}(iT,z)\,,\\
{}^{(\mathrm{R})}\langle\langle \ell,m|\, e^{-\pi T H^{(c)}} e^{i\pi z(J_0+\tilde{J}_0)}\, |\ell',m'\rangle\rangle^{(\mathrm{R})} &= \left(\delta_{\ell,\ell'}\delta_{m,m'} + \delta_{\ell,k-\ell'}\delta_{m,m'+k+2}\right) \times\\
&\quad \times \mathrm{ch}^{(\mathrm{R})}_{\ell,m}(iT,z)\,,
\end{aligned} \tag{4.3}$$

where $\mathrm{ch}^{(\mathrm{NS})}_{\ell,m}(\tau,z)$ ($\mathrm{ch}^{(\mathrm{R})}_{\ell,m}(\tau,z)$) denotes the NS (R) character of the $\mathcal{N}=2$ minimal model with the primary field with $h = \ell(\ell+2) - m^2/4(k+2)$, $Q = m/k+2$ ($h = \frac{\ell(\ell+2)-m^2}{4(k+2)} + \frac{1}{8}$, $Q = \frac{m}{k+2} \pm \frac{1}{2}$). (See appendix A.) Note that the character identities

$$\begin{aligned}
\mathrm{ch}^{(\sigma)}_{k-\ell,m+k+2}(\tau,z) &= \mathrm{ch}^{(\sigma)}_{\ell,m}(\tau,z)\,, \qquad (\sigma = \mathrm{NS},\, \mathrm{R})\,,\\
\mathrm{ch}^{(\sigma)}_{k-\ell,m+k+2}(\tau,z) &= -\mathrm{ch}^{(\sigma)}_{\ell,m}(\tau,z)\,, \qquad (\sigma = \widetilde{\mathrm{NS}},\, \widetilde{\mathrm{R}})\,,
\end{aligned} \tag{4.4}$$

imply the field identification

$$|k-\ell, m+k+2\rangle\rangle^{(\sigma)} = |\ell,m\rangle\rangle^{(\sigma)}\,. \tag{4.5}$$

We can thus restrict ourselves to the range $0 \leq \ell \leq k$, $m \in \mathbb{Z}_{k+2}$ without loss of generality. It is also convenient to set $|\ell,m\rangle\rangle^{(\mathrm{NS})} = 0$ ($|\ell,m\rangle\rangle^{(\mathrm{R})} = 0$), if $\ell+m \in 2\mathbb{Z}+1$ ($\ell+m \in 2\mathbb{Z}$). Then, the Cardy states are expressed as follows ($\sigma = \mathrm{NS}$, or R);

$$\begin{aligned}
&|L,M\rangle^{(\sigma)} = \sum_{\ell=0}^{k} \sum_{m\in\mathbb{Z}_{k+2}} C_{L,M}(\ell,m)\, |\ell,m\rangle\rangle^{(\sigma)}\,, \qquad C_{L,M}(\ell,m) = \frac{S^{L,M}_{\ell,m}}{\sqrt{S^{0,0}_{\ell,m}}}\,,\\
&L+M \in 2\mathbb{Z} \qquad (\text{for } \sigma = \mathrm{NS})\,, \qquad L+M \in 2\mathbb{Z}+1 \qquad (\text{for } \sigma = \mathrm{R})\,,
\end{aligned} \tag{4.6}$$

where $S^{\ell,m}_{\ell',m'}$ is the modular coefficients of $\mathrm{ch}^{(\mathrm{NS})}_{\ell,m}(\tau,z)$, given explicitly by

$$S^{\ell,m}_{\ell',m'} = 2\cdot\sqrt{\frac{2}{k+2}}\sin\left(\frac{\pi(\ell+1)(\ell'+1)}{k+2}\right)\cdot\frac{1}{\sqrt{2(k+2)}} e^{2\pi i \frac{mm'}{2(k+2)}}\,. \tag{4.7}$$

(The overall factor 2 is due to the choice of range $m \in \mathbb{Z}_{k+2}$ instead of $\mathbb{Z}_{2(k+2)}$.)

All the quantities in R-sector are generated by the 1/2-spectral flow $U_{1/2}$ ($\eta = 1/2$ in (2.6)) from those of NS-sector. Then, we find

$$|\ell, m\rangle\rangle^{(\mathrm{R})} = U_{1/2}\,|\ell, m+1\rangle\rangle^{(\mathrm{NS})}\,, \qquad |L, M\rangle^{(\mathrm{R})} = e^{-i\pi\frac{M}{k+2}}U_{1/2}e^{-i\pi\frac{1}{2}(J_0+\tilde{J}_0)}|L, M+1\rangle^{(\mathrm{NS})}\,. \tag{4.8}$$

Let us next consider the Liouville sector. Our aim is to identify the BPS D-branes wrapped around the vanishing cycles. As we already discussed, the class 1 Cardy states $|B; r\rangle$ correspond to the D-branes localized in the strong coupling region $\phi \sim +\infty$, which is near the isolated singularity in the Calabi-Yau space in this context. Furthermore, the overlaps among the class 1 states only contains the open strings of discrete spectrum, implying compact world-volumes of D-branes. The class 2 and 3 states do not have such a property. From these observations, it is plausible to guess that the class 1 states capture the open string excitations moving near the singularity and characterize the geometry of the singular Calabi-Yau space. Therefore, we propose that.

Supersymmetric boundary states describing the vanishing cycles are given by the class 1 states of $\mathcal{N} = 2$ Liouville theory.

Using the 1/2-spectral flow, the extended characters for the graviton representation (2.22) for other spin structures are given by

$$\begin{aligned}
\chi_{\mathbf{G}}^{(\widetilde{\mathrm{NS}})}(r;\tau,z) &= e^{-i\pi\frac{2Kr}{N}}\,\chi_{\mathbf{G}}^{(\mathrm{NS})}\left(r;\tau,z+\frac{1}{2}\right), \qquad (r\in\mathbb{Z}_N)\,,\\
\chi_{\mathbf{G}}^{(\mathrm{R})}(r;\tau,z) &= q^{\hat{c}/8}y^{\hat{c}/2}\,\chi_{\mathbf{G}}^{(\mathrm{NS})}\left(r-\frac{1}{2};\tau,z+\frac{\tau}{2}\right), \qquad \left(r\in\frac{1}{2}+\mathbb{Z}_N\right),\\
\chi_{\mathbf{G}}^{(\widetilde{\mathrm{R}})}(r;\tau,z) &= e^{-i\pi\frac{K(2r-1)}{N}}q^{\hat{c}/8}y^{\hat{c}/2}\,\chi_{\mathbf{G}}^{(\mathrm{NS})}\left(r-\frac{1}{2};\tau,z+\frac{\tau}{2}+\frac{1}{2}\right), \qquad \left(r\in\frac{1}{2}+\mathbb{Z}_N\right),
\end{aligned} \tag{4.9}$$

and we define

$$\begin{aligned}
|p,j\rangle\rangle^{(\mathrm{R})} &= U_{1/2}\,|p,j-K\rangle\rangle^{(\mathrm{NS})}\,, \quad |r,s\rangle\rangle_{\mathbf{M}}{}^{(\mathrm{R})} = U_{1/2}\left|r-\frac{1}{2},s\right\rangle\!\!\right\rangle^{(\mathrm{NS})}, \quad \left(r\in\frac{1}{2}+\mathbb{Z}_N\right)\\
|B;r\rangle^{(\mathrm{R})} &= e^{-i\pi\frac{K(2r-1)}{N}}U_{1/2}e^{-\frac{i\pi}{2}(J_0+\tilde{J}_0)}\left|B;r-\frac{1}{2}\right\rangle^{(\mathrm{NS})}, \quad \left(r\in\frac{1}{2}+\mathbb{Z}_N\right),
\end{aligned} \tag{4.10}$$

similarly to (4.8).

Now, taking account of the GSO projection, the proposed boundary states have the form

$$\sqrt{N}\,P_{\mathrm{GSO}}\left(|L,M\rangle^{(\sigma)}\otimes|B;r\rangle^{(\sigma)}\right), \tag{4.11}$$

where P_{GSO} stands for the GSO projection operator imposing the integrality of total U(1)-charge in the closed string Hilbert space. $\sqrt{N}$ is a normalization constant fixed by the Cardy condition as we shall see below.

$P_{\rm GSO}$ restricts the choice of the tensor products of Ishibashi states to;

- NS-sector:

$$
\begin{aligned}
&|\ell,m\rangle\rangle^{(\rm NS)}\otimes|r,s\rangle\rangle_{\mathbf{M}}{}^{(\rm NS)}\,, && \frac{m}{k+2}+\frac{s+2Kr}{N}\in\mathbb{Z}\,, \qquad (r\in\mathbb{Z}_N)\,,\\
&|\ell,m\rangle\rangle^{(\rm NS)}\otimes|p,j\rangle\rangle^{(\rm NS)}\,, && \frac{m}{k+2}+\frac{j}{N}\in\mathbb{Z}\,.
\end{aligned} \tag{4.12}
$$

- R-sector:

$$
\begin{aligned}
|\ell,m\rangle\rangle^{(\rm R)}\otimes|r,s\rangle\rangle_{\mathbf{M}}{}^{(\rm R)}\,, \quad & \frac{m}{k+2}+\frac{s+2Kr}{N}\in\mathbb{Z}\,, \quad \left(r\in\frac{1}{2}+\mathbb{Z}_N\right), \quad \text{for } d=2,6\,,\\
& \frac{m}{k+2}+\frac{s+2Kr}{N}-\frac{1}{2}\in\mathbb{Z}\,, \quad \left(r\in\frac{1}{2}+\mathbb{Z}_N\right), \quad \text{for } d=4\,,\\
|\ell,m\rangle\rangle^{(\rm R)}\otimes|p,j\rangle\rangle^{(\rm R)}\,, \quad & \frac{m}{k+2}+\frac{j}{N}\in\mathbb{Z}\,, \quad \text{for } d=2,6\,,\\
& \frac{m}{k+2}+\frac{j}{N}-\frac{1}{2}\in\mathbb{Z}\,, \quad \text{for } d=4\,,
\end{aligned} \tag{4.13}
$$

From (3.36) it is easy to find that (4.11) depends on the parameters M, r only through the combination $M+2r$. We can thus simply set $r=0$ ($r=1/2$) for the NS (R) sector (namely, $|B;O\rangle$ itself for the NS-sector) without loss of generality. In this way we consider the boundary states

$$
\begin{aligned}
|B;L,M\rangle^{(\rm NS)} &= \sqrt{N}\,P_{\rm GSO}\left(|L,M\rangle^{(\rm NS)}\otimes|B;r=0\rangle^{(\rm NS)}\right),\\
|B;L,M\rangle^{(\rm R)} &= \sqrt{N}\,P_{\rm GSO}\left(|L,M\rangle^{(\rm R)}\otimes\left|B;r=\frac{1}{2}\right\rangle^{(\rm R)}\right),
\end{aligned} \tag{4.14}
$$

and set

$$
|B;L,M,\pm\rangle=|B;L,M\rangle^{(\rm NS)}\pm|B;L,M-1\rangle^{(\rm R)}\,, \qquad (L+M\in 2\mathbb{Z})\,. \tag{4.15}
$$

Obviously $|B;L,M,+\rangle$ and $|B;L,M,-\rangle$ can be interpreted as the brane and anti-brane respectively. We also note

$$
|B;k-L,M+k+2\rangle^{(\rm NS)}=|B;L,M\rangle^{(\rm NS)}\,, \quad |B;k-L,M+k+2\rangle^{(\rm R)}=-|B;L,M\rangle^{(\rm R)}\,, \tag{4.16}
$$

and thus

$$
|B;k-L,M+k+2,\pm\rangle=|B;L,M,\mp\rangle\,. \tag{4.17}
$$

Therefore, it is enough to restrict ourselves to the range $L=0,1,\dots,[k/2]$, $M\in\mathbb{Z}_{2N}$, as long as we consider only the branes (or the anti-branes).

Let us analyse the overlap amplitudes of the total boundary states (4.14). We first consider the NS sector. Presence of $P_{\rm GSO}$ is the only non-trivial point in this calculation. The easiest way to handle it is to insert

$$
\delta^{(N)}\left(\frac{N}{k+2}m+\beta\right)\equiv\frac{1}{N}\sum_{r\in\mathbb{Z}_N}e^{-2i\pi r\left(\frac{1}{k+2}m+\frac{1}{N}\beta\right)}\,, \qquad (\beta=j \quad \text{or} \quad s+2Kr)\,, \tag{4.18}
$$

into the amplitudes. We obtain after a little calculation ($\hat{c} = 5 - d/2$);

$$e^{\pi \frac{\hat{c} z^2}{T}} {}^{(\mathrm{NS})}\langle L_1, M_1 | e^{-\pi T H^{(c)}} e^{i\pi z (J_0 + \tilde{J}_0)} | L_2, M_2 \rangle^{(\mathrm{NS})}$$
$$= \sum_{L=0}^{k} \sum_{r \in \mathbb{Z}_N} N_{L_1,L_2}^{L} \, \mathrm{ch}_{L, M_2 - M_1 - 2r}^{(\mathrm{NS})}(it, z') \chi_{\mathbf{G}}^{(\mathrm{NS})}(r; it, z') \,, \tag{4.19}$$

where N_{L_1,L_2}^{L} are the familiar fusion coefficients of $\mathrm{SU}(2)_k$;

$$N_{L_1,L_2}^{L} = \begin{cases} 1 & |L_1 - L_2| \le L \le \min[L_1 + L_2, 2k - L_1 - L_2]\,, \quad L \equiv |L_1 - L_2| \pmod 2) \\ 0 & \text{otherwise} \end{cases} \tag{4.20}$$

Normalization factor $\sqrt{N}$ of the boundary state has been chosen so that the factor 1 appears in front of the r.h.s. of (4.19).

The overlap (4.19) implies that the open strings have integral U(1)-charges, if and only if $M_1 \equiv M_2 \pmod{k+2}$. Careful examination of other spin structures leads to the result: *the brane configurations are supersymmetric if and only if* $M_1 \equiv M_2 \pmod{2(k+2)}$. (The cases of $M_1 \equiv M_2 + (k+2) \pmod{2(k+2)}$ can be reinterpreted as the D-$\bar{D}$ systems.) Similar observation has been given in [58] in the context of supersymmetric $\mathrm{SU}(2)_k$ WZW model in the $d = 6$ case. When $M_1 \equiv M_2 \pmod{2(k+2)}$ holds, the cancellation of cylinder amplitude due to the space-time SUSY takes place

$$\sum_{\sigma} \sum_{L=0}^{k} \sum_{r \in \mathbb{Z}_N + \alpha(\sigma)} \epsilon(\sigma) \left(\frac{\theta_{[\sigma]}(\tau, 0)}{\eta(\tau)} \right)^{\frac{d-2}{2}} N_{L_1,L_2}^{L} \mathrm{ch}_{L,-2r}^{(\sigma)}(it, 0) \chi_{\mathbf{G}}^{(\sigma)}(r; it, 0) = 0 \,. \tag{4.21}$$

Here $(\theta_{[\sigma]}(\tau,0)/\eta(\tau))^{(d-2)/2}$ part comes from the transverse space-time $\mathbb{R}^{d-2}$ and we set $\epsilon = +1$ for $\sigma = \mathrm{NS}, \widetilde{\mathrm{R}}$, $\epsilon = -1$ for $\sigma = \widetilde{\mathrm{NS}}, \mathrm{R}$, $\alpha = 0$ for $\sigma = \mathrm{NS}, \widetilde{\mathrm{NS}}$, $\alpha = 1/2$ for $\sigma = \mathrm{R}, \widetilde{\mathrm{R}}$, and $\theta_{[\mathrm{NS}]} = \theta_3$, $\theta_{[\widetilde{\mathrm{NS}}]} = \theta_4$, $\theta_{[\mathrm{R}]} = \theta_2$, $\theta_{[\widetilde{\mathrm{R}}]} = i\theta_1$. The above relation can be proven as a special case of SUSY cancellation of the $\hat{c} = 4$ extended characters shown in [47]. Thus the self-overlaps of $|B; L, M, \pm\rangle$ are supersymmetric for arbitrary L, M and we conclude that they are indeed BPS $D(\bar{D})$-branes.

Now let us identify the boundary states $|B; L, M, \pm\rangle$ with the vanishing cycles in a singular Calabi-Yau manifold. In the case of A-type singularity vanishing cycles can be represented by line segments connecting pairs of roots of $X^{k+2} = \mu$ in the complex X-plane (see [20], for instance). Here the deformation parameter μ is identified with the cosmological constant associated with the operator S_+ (2.1). We parameterize pairs of roots as

$$X_{L,M,+} = \mu^{1/(k+2)} e^{i\pi(M+L)/(k+2)}\,, \qquad X_{L,M,-} = \mu^{1/(k+2)} e^{i\pi(M-L-2)/(k+2)}\,, \tag{4.22}$$
$$L = 0, 1, \ldots, \left[\frac{k}{2}\right], \qquad M \in \mathbb{Z}_{2(k+2)}\,, \qquad L + M \in 2\mathbb{Z}\,, \tag{4.23}$$

and denote the corresponding cycle as $\gamma_{L,M}$. As has been often pointed out in the literature (see, [18, 20] for instance), parameters L, M of the vanishing cycles are also identified as quantum numbers L and M of the minimal model themselves. We thus claim the correspondence

$$|B;L,M,\pm\rangle \longleftrightarrow D\ (\bar{D}) \text{ brane wrapped around } \gamma_{L,M}\,. \tag{4.24}$$

We here emphasize that the description of the Liouville sector of the boundary states $|B;L,M,\pm\rangle$ is the essential new ingredient of this work.

As an important consistency check let us evaluate the open string Witten indices, which are identified as the intersection numbers of vanishing cycles. The evaluation is almost parallel to that of the overlap (4.19), but in place of (4.18) we have to insert a factor

$$\begin{aligned}\delta^{(2N)}\left(\frac{N}{k+2}m+\beta-\frac{N\epsilon}{2}\right)-\delta^{(2N)}\left(\frac{N}{k+2}m+\beta+N-\frac{N\epsilon}{2}\right)\equiv\\ \equiv\frac{1}{N}\sum_{r\in\frac{1}{2}+\mathbb{Z}_N}e^{-2i\pi r\left(\frac{1}{k+2}m+\frac{1}{N}\beta\right)+i\pi\epsilon r}\,,\qquad(\beta=j\quad\text{or}\quad s+2Kr)\,,\end{aligned}\tag{4.25}$$

where $\epsilon=0$ for $d=2,6$ and $\epsilon=1$ for $d=4$. The relative minus sign in the left-hand-side of this equation is due to the insertion of $e^{\frac{i\pi}{2}(F+\tilde{F})}$, where F and $\tilde{F}$ denote the world-sheet fermion number operators. Open string Witten indices are thus evaluated as

$$\begin{aligned}I(L_1,M_1|L_2,M_2)&\equiv{}^{(\mathrm{R})}\langle B;L_1,M_1,+|e^{-\pi TH^{(c)}}e^{\frac{i\pi}{2}(F+\tilde{F})}|B;L_2,M_2,+\rangle^{(\mathrm{R})}\\ &=-\sum_{L=0}^{k}\sum_{r\in\frac{1}{2}+\mathbb{Z}_N}e^{i\pi\epsilon r}N^{L}_{L_1,L_2}\,\mathrm{ch}^{(\widetilde{\mathrm{R}})}_{L,M_2-M_1-2r}(it,0)\chi^{(\widetilde{\mathrm{R}})}_{\mathbf{G}}(r;it,0)\,.\end{aligned}\tag{4.26}$$

To proceed further, we recall the Witten index for the minimal model;

$$\mathrm{ch}^{(\widetilde{\mathrm{R}})}_{\ell,m}(\tau,0)=\delta^{(2(k+2))}(m-(\ell+1))-\delta^{(2(k+2))}(m+(\ell+1))\,. \tag{4.27}$$

On the other hand in the Liouville sector we have a formula

$$\chi^{(\widetilde{\mathrm{R}})}_{\mathbf{G}}(r;\tau,0)=\delta^{(N)}\left(r-\frac{1}{2}\right)-\delta^{(N)}\left(r+\frac{1}{2}\right)\,, \tag{4.28}$$

derived from (2.22) and the definition of $\widetilde{\mathrm{R}}$-character (4.9).[7] We finally obtain

$$\begin{aligned}I(L_1,M_1|L_2,M_2)=&\sum_{L=0}^{k}\sum_{a_i=\pm1}\mathrm{sgn}(a_1)\mathrm{sgn}(a_2)e^{\frac{i\pi}{2}\epsilon a_2}N^{L}_{L_1,L_2}\delta^{(2(k+2))}\times\\ &\times(M_1-M_2-a_1(L+1)+a_2)\,.\end{aligned}\tag{4.29}$$

[7]The easiest way to derive (4.28) is to recall the character identity

$$\begin{aligned}\chi^{(\widetilde{\mathrm{R}})}_{\mathbf{G}}(r;\tau,z)&=q^{-\frac{K}{4N}}\Theta_{2Kr,NK}\left(\tau,\frac{2z}{N}\right)\frac{i\theta_1(\tau,z)}{\eta(\tau)^3}+\chi^{(\widetilde{\mathrm{R}})}_{\mathbf{M}}(r,N;\tau,z)-\chi^{(\widetilde{\mathrm{R}})}_{\mathbf{M}}(r-1,2K;\tau,z)\,,\\ \chi^{(\widetilde{\mathrm{R}})}_{\mathbf{M}}(r,s;\tau,z)&\equiv e^{-i\pi\frac{s+K(2r-1)}{N}}q^{\frac{\hat{c}}{8}}y^{\frac{\hat{c}}{2}}\chi^{(\mathrm{NS})}_{\mathbf{M}}\left(r-\frac{1}{2},s;\tau,z+\frac{\tau}{2}+\frac{1}{2}\right)\\ &=\sum_{m\in\mathbb{Z}}\frac{\left(yq^{N\left(m+\frac{2r+1}{2N}\right)}\right)^{\frac{s-K}{N}}}{1-yq^{N\left(m+\frac{2r+1}{2N}\right)}}y^{2K\left(m+\frac{2r+1}{2N}\right)}q^{NK\left(m+\frac{2r+1}{2N}\right)^2}\frac{i\theta_1(\tau,z)}{\eta(\tau)^3}\,,\end{aligned}$$

which is shown by taking the 1/2-spectral flow of (2.24).

This is the desired result which reproduce the correct intersection numbers of vanishing cycles $\gamma_{L,M+1}$. In particular, we find for the "fundamental cycles" $\gamma_{0,M+1}$,

$$I(0,M_1|0,M_2) = 2\delta^{(2(k+2))}(M_1-M_2) - \delta^{(2(k+2))}(M_1-M_2-2) - \delta^{(2(k+2))}(M_1-M_2+2)\,, \tag{4.30}$$

for $d=2,6$ and

$$I(0,M_1|0,M_2) = \delta^{(2(k+2))}(M_1-M_2-2) - \delta^{(2(k+2))}(M_1-M_2+2)\ , \tag{4.31}$$

for $d=4$.

Note that (4.30) agrees with the A_N type extended Cartan matrix while (4.31) is anti-symmetric in M_1 and M_2 as expected for Langrangian 3-cycles.

5. D-branes in type-0 string vacua based on the $\mathcal{N}=2$ Liouville with $\hat{c}=5$

Let us next discuss the type-0 string vacua constructed from the $\mathcal{N}=2$ Liouville theory, motivated by the recent studies on the duality between matrix models and non-critical strings, and unstable D-branes [21]–[41]. We hence only focus on the case $\hat{c}=5$ ($\mathcal{Q}=2$) which describe a two dimensional target space. Generalization to other values of $\hat{c}$ is straightforward.

The type-0 string vacua are defined by the diagonal GSO projection. Since they are non-supersymmetric, we need not impose the integrality of U(1)-charge. We may compactify along the Y-direction with discretized radius $R=M\mathcal{Q}\equiv 2M$ ($M\in\mathbb{Z}_{>0}$, in the $\alpha'=2$ unit) by imposing the locality of the Liouville potential terms S_+, S_-. The minimal radius $R=2$ ($M=1$) corresponds to the integral U(1)-charges and fits to the type-II string vacua. More general radius $R=2M$ corresponds to $N=M$, $K=2M$ and yields fractional U(1)-charges $Q=n/M\,, n\in\mathbf{Z}$. Since we have no transverse degrees of freedom, the Y-direction is naturally identified with the euclidean time. In this sense the system we are considering may be regarded as a thermal model with discretized temperatures $T=1/(4\pi M)$, analogous to the thermal S-brane models [59] (see also, e.g. [60]–[63]).

The type-0 GSO projections are given by

- **type 0A** : $J_0+\tilde{J}_0\in 2\mathbb{Z}$,
- **type 0B** : $J_0-\tilde{J}_0\in 2\mathbb{Z}$.

We now concentrate on the A and B-type boundary states in the type-0B theory. The B and A-type branes in the type-0A theory can be studied in a parallel way.

A-branes: the A-branes in the type-0B theory are quite easy. The class 1 A-branes should be the D-instantons and the class 2, 3 A-branes correspond to the $D0$-instanton (Dirichlet along the Y-direction, and Neumann along ϕ-direction). All we have to do in the calculation of cylinder amplitudes is to set $N=M$, $K=2M$ rather than $N=1$, $K=2$ in our analyses. We denote the Cardy states for the A-branes as

$$|B;*,\pm\rangle_{\mathrm{A}} = |B;*\rangle^{(\mathrm{NS})}_{\mathrm{A}} \pm |B;*\rangle^{(\mathrm{R})}_{\mathrm{A}}\,. \tag{5.1}$$

For example, for the class 1 branes we obtain

$$
\begin{aligned}
{}_{\mathrm{A}}\langle B; r, + | e^{-\pi T H^{(c)}} | B; r', + \rangle_{\mathrm{A}} = \; & \chi^{(\mathrm{NS})}_{\mathbf{G},\, N=M, K=2M}(r'-r; it, 0) - \\
& -\chi^{(\widetilde{\mathrm{NS}})}_{\mathbf{G},\, N=M, K=2M}(r'-r; it, 0)\,, \qquad (r, r' \in \mathbb{Z}_M)\,. \qquad (5.2)
\end{aligned}
$$

The $\widetilde{\mathrm{NS}}$-contribution in the open string channel originates from the RR-part of the Cardy state. The oscillator part θ_3/η^3 in the extended characters is canceled by the contributions from superconformal ghosts. It is obvious that r expresses the position of D-instanton along the S^1 of Y-direction.

Note also that

$$
{}_{\mathrm{A}}\langle B; r, - | e^{-\pi T H^{(c)}} | B; r', + \rangle_{\mathrm{A}} = \chi^{(\mathrm{NS})}_{\mathbf{G}, N=M, K=2M}(r'-r; it, 0) + \chi^{(\widetilde{\mathrm{NS}})}_{\mathbf{G}, N=M, K=2M}(r'-r; it, 0)\,, \qquad (5.3)
$$

which clearly includes the tachyon mode when $r = r'$. $|B; r, -\rangle_{\mathrm{A}}$ is of course identified as the $\bar{D}$ instanton. $|B; *, -\rangle_{\mathrm{A}}$ for the class 2 and 3 branes are similarly interpreted as $\bar{D}1$-branes.

The calculations for the class 2 and 3 branes are similarly carried out, and in all the cases of D-D cylinder amplitudes, we find only the GSO projected[8] NS open strings and no R open strings. The open string vacuum states have conformal weights $h \geq 1/2$, implying that these D-branes are stable. However, the D-$\bar{D}$ systems could be unstable as in the class 1 cases. We specially need a careful analysis to examine the stability of the class 3 D-$\bar{D}$ systems for general M.

Under the decompactification limit $M \to \infty$, we have the continuous spectrum of U(1)-charges and the extended characters are reduced to the irreducible ones (2.3), (2.4) and (2.5), as we already noted. For the modular bootstrap we should use the formulas (2.33), (2.34) and (2.35). The analysis is almost parallel and the Cardy states are defined associated to the irreducible representations $\mathrm{ch}^{(*)}(h, j_0, n; \tau, z)$, $\mathrm{ch}^{(*)}_{\mathbf{M}}(\lambda, n; \tau, z)$, $\mathrm{ch}^{(*)}_{\mathbf{G}}(n; \tau, z)$ of section 2. There is one difference from the discrete case: we can consistently define the class 3 states associated with each of the massless matter representations $\mathrm{ch}^{(*)}_{\mathbf{M}}(\lambda, n; \tau, z)$. This is because the S-transformation formula (2.34) does not contain the "boundary term" contrary to (2.30). The cylinder amplitudes are similarly evaluated. For example, we have

$$
{}_{\mathrm{A}}\langle B; n, + | e^{-\pi T H^{(c)}} | B; n', + \rangle_{\mathrm{A}} = \mathrm{ch}^{(\mathrm{NS})}_{\mathbf{G}}(n'-n; it, 0) - \mathrm{ch}^{(\widetilde{\mathrm{NS}})}_{\mathbf{G}}(n'-n; it, 0)\,, \qquad (n, n' \in \mathbb{Z})\,, \qquad (5.4)
$$

for the class 1 branes.

B-branes: the B-branes in the type-0B theory is more non-trivial. We first consider the theory of the minimal radius $R = 2$ ($M = 1$). In this case U(1)-charge for the NSNS (RR) Ishibashi states are integral (half odd integral). On the other hand, the type-0B GSO

[8]Here the "GSO projected" precisely means that the open string U(1)-charge satisfies

$$
J_0 \in Q' - Q + 2\mathbb{Z} + 1\,,
$$

where Q, Q' are the U(1)-charges of the representations labeling the Cardy states.

projection forces all the B-type-Ishibashi states to have integral charges. Thus the GSO projection eliminates all the RR-sector of the theory. We thus obtain the general Cardy states for the B-branes $|B;*\rangle_{\mathrm{B},\,R=2}$ constructed from the B-type-Ishibashi states only of the NSNS sector. The B-branes so constructed possess no RR-charges and are regarded as the generalizations of the non-BPS D-branes in the flat space-time (see [64], for instance). The class 1 B-branes should be $D0$-branes and the class 2, 3 B-branes should be $D1$-branes.

The cylinder amplitudes only contain the NS open strings without the GSO projection, as in the non-BPS D-branes in the flat background. We have the unique class 1 state $|B;r=0\rangle_{\mathrm{B},\,R=2}$, and its overlap is evaluated as

$$ {}_{R=2,\,\mathrm{B}}\langle B;0|e^{-\pi T H^{(c)}}|B;0\rangle_{\mathrm{B},\,R=2} = \chi^{(\mathrm{NS})}_{\mathbf{G},\,N=1,K=2}(r=0;it,0)\,. \tag{5.5} $$

This is tachyonic[9] as is expected. On the other hand, all the class 2, 3 branes contain no tachyonic open string states despite the lack of open string GSO projection (under the assumption (3.57) for the class 2-branes) because of the mass gap $\mathcal{Q}^2/8=1/2$. See appendix D for the class 3 branes.

The Cardy states for general radius $R=2M$ are obtained by making the "thermal projection operator" P_M act on $|B;*\rangle_{\mathrm{B},\,R=2}$, as in the construction of the S-brane boundary states at finite temperature [59];

$$ |B;*\rangle_{\mathrm{B},\,R=2M} = \mathcal{C}P_M|B;*\rangle_{\mathrm{B},\,R=2}\,. \tag{5.6} $$

$\mathcal{C}$ is a normalization constant to be fixed by the Cardy condition and P_M is the projection operator to the sectors generated by the vacua with the following spectrum of U(1)-charges

$$ J_0 = \frac{n}{M}+2Mw\,, \qquad \tilde{J}_0 = \frac{n}{M}-2Mw\,, \qquad (n,w\in\mathbb{Z})\,. \tag{5.7} $$

We can calculate their overlaps in a manner similar to [65]: thanks to the B-type boundary condition (Neumann along the Y-direction), one may replace the projection P_M with a simpler one

$$ \frac{1}{M}\sum_{s\in\mathbb{Z}_M} e^{i\pi\frac{s}{M}(J_0-\tilde{J}_0)}\,, \tag{5.8} $$

in the amplitudes. We obtain for the class 1 brane, for example,

$$ \begin{aligned} {}_{R=2M,\mathrm{B}}\langle B;0|e^{-\pi T H^{(c)}}|B;0\rangle_{\mathrm{B},R=2M} &= \frac{|\mathcal{C}|^2}{M}\sum_{s\in\mathbb{Z}_M} e^{-2\pi t\cdot\frac{5}{2}\left(\frac{s}{M}\right)^2}\chi^{(\mathrm{NS})}_{\mathbf{G},\,N=1,K=2}\times \\ &\quad\times\left(r=0;it,-it\left(\frac{s}{M}\right)\right)\,. \end{aligned} \tag{5.9} $$

[9] Since we are now working in the euclidean theory with no transverse degrees of freedom, we do not have the mass shell condition in the usual sense. We look at the IR behavior of the open string amplitudes (after making the oscillator part canceled out with the ghost sector)

$$ \chi(it)\sim e^{-2\pi t m^2}\,, \qquad (t\to\infty)\,, $$

and regard it as massive if $m^2>0$, massless if $m^2=0$, and tachyonic if $m^2<0$.

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Thus we should choose $\mathcal{C} = \sqrt{M}$. One can see that the character function in the open string channel is generated by a fractional spectral flow with $\eta = s/M$ in (2.6). This is because the projection (5.8) works as twistings along the spatial direction in the open string channel under the modular transformation. We note that, even if we have closed string states with only integral U(1)-charges, fractional U(1)-charges may appear in the open string spectrum. We again find an open string tachyon in (5.9).

In the limit $M \to \infty$ we instead obtain

$$\begin{aligned} {}_{R=\infty,\,\mathrm{B}}\langle B;0|e^{-\pi T H^{(c)}}|B;0\rangle_{\mathrm{B},\,R=\infty} &= |\mathcal{C}|^2 \int_0^1 dx\, e^{-2\pi t\cdot\frac{5}{2}x^2} \chi^{(\mathrm{NS})}_{\mathbf{G},\,N=1,K=2}(r=0;it,-itx) \\ &\equiv |\mathcal{C}|^2 \int_{-\infty}^{\infty} dx\, e^{-2\pi t\cdot\frac{5}{2}x^2} \mathrm{ch}^{(\mathrm{NS})}_{\mathbf{G}}(it,-itx)\,, \end{aligned} \tag{5.10}$$

and hence we must set $\mathcal{C} = 1$ in this case. The spectral flow parameter becomes continuous in this decompactification limit, and this behavior is consistent with the fact that the U(1)-direction (euclidean time) is the Neumann boundary condition for the B-branes.

The calculations for the class 2 and 3 branes are similar. All we have to do is to perform a sum of the results for $R = 2$ over the fractional spectral flows $U_{s/M}$. The stabilities of these branes again follow from the mass gap.

In conclusion, the class 1 B-branes are identified as the unstable $D0$-branes, which presumably possess a matrix model description and the class 2 and 3 B-branes are the $D1$-branes, which are stable due to the mass gap.

We would like to present several remarks.

1. Let us briefly discuss the type-IIB case. (Similar results in the IIA case are obtained by exchanging the roles of A and B-branes.) This time only the minimal radius $R = 2$ is allowed because of the locality of space-time supercharges. The A-branes are of course the BPS branes we treated in the previous sections, while the B-branes are analogues of the non-BPS branes in the flat background. The boundary states for the IIB B-branes are the same as the 0B B-branes given above, but we must insert the GSO projection $\frac{1+(-1)^F}{2}$ in the overlap amplitudes. The net effect amounts to providing the R-sector in the open string channel. For example, we obtain for the unique class 1 state,[10]

$$\begin{aligned} 2 \times {}_{R=2,\,\mathrm{B}}\langle B;0|e^{-\pi T H^{(c)}}\frac{1+(-1)^F}{2}|B;0\rangle_{\mathrm{B},\,R=2} &= \\ = \chi^{(\mathrm{NS})}_{\mathbf{G},\,N=1,K=2}(r=0;it,0) - \chi^{(\mathrm{R})}_{\mathbf{G},\,N=1,K=2}(r=1;it,0)\,, \end{aligned} \tag{5.11}$$

where we included the factor $(\sqrt{2})^2 \equiv 2$ due to the tension of non-BPS D-branes. The amplitude includes the tachyon mode in the NS sector and the massless mode in the R sector. The class 1 B-branes are identified as the non-BPS $D0$-branes in the IIB vacuum. Based on the above analysis on the overlaps, it seems plausible that

[10]Precisely speaking, when calculating thermal amplitudes, we need to take care of the boundary condition for space-time fermions along the thermal circle (Y-direction) (see, for instance, [66]). We shall here omit it to avoid unessential complexity.

these unstable branes possess a supermatrix model description. It has been recently proposed in [31] that the IIB string vacuum of the $\mathcal{N}=2$ Liouville theory should be dual to the supermatrix quantum mechanics introduced by Marinari and Parisi [67].[11]

Note also that such type-II vacua have no bose-fermi cancellation in space-time, although the space-time supercharges are still well-defined as local operators. This is because the two dimensional Poincaré invariance is broken due to the Liouville potentials, as is pointed out in [33]. Interestingly, the space-time supercharges possess nilpotency; $Q^2=0$ and are interpreted as some BRST charges, suggesting a topological symmetry behind the system [68].

2. The 2d type-0 model considered here has the same field content as that of the $c=3/2$ $\mathcal{N}=1$ superconformal matter coupled to the $\mathcal{N}=1$ Liouville theory, studied recently in detail in the context of dual matrix model [26, 27, 32, 35, 39, 41]. However, one should remember that the $\mathcal{N}=2$ Liouville potential $\mu S_+ + \bar{\mu} S_-$ is the "sine-Liouville type" [69]. It is analogous to the inhomogeneous brane decay/creation models considered in e.g. [70]. However, an important difference is that our system is already time dependent *in the bulk*, because the $\mathcal{N}=2$ Liouville potential breaks the translational invariance $Y \to Y+a$.

3. The $\mathcal{N}=2$ Liouville theory with the minimal radius $R=\mathcal{Q}(=2)$ is known to be T-dual to the $\mathrm{SL}(2;\mathbb{R})/\mathrm{U}(1)$ Kazama-Suzuki model [11, 71] (see also [69, 72]). In a recent paper [33], it has been conjectured that the 2d type-0 model considered here with $R=2$ should be dual to the $\mathbb{Z}_2$-symmetric version of the KKK matrix model at the self-dual radius [69]. This conjecture is inspired by the proposal for the $\mathcal{N}=1$ case given in [26, 27], and based on the similarity of $\mathcal{N}=2$ Liouville to the sine-Liouville theory. It may be interesting to study the models with $R=2M$ and to explore the relations to the "symmetric KKK matrix models" with general radii.

6. Summary and discussions

In this paper we have investigated the boundary states in the $\mathcal{N}=2$ Liouville theory with arbitrary rational central charge $\hat{c}=1+2K/N$ based on the modular bootstrap approach. The key idea is to take a sum over spectral flows in order to yield discrete spectra of U(1)-charges both in the closed and open string channels. This property is necessary to construct the BPS D-branes in superstring vacua. Our modular bootstrap approach is able to determine only the absolute values of boundary wave functions and we have fixed their phase factors by making use the proposal of reflection amplitudes given in [52, 53]. These boundary wave functions show several nice features for their identifications as localized/extended D-branes. In any case, we emphasize that all the informations necessary for the derivation of cylinder amplitudes can be obtained by our method, and we can

[11]However, our boundary state of the class 1 B-brane, which we believe to describe the non-BPS $D0$-brane, differs from the one proposed in [31]. The latter includes the Ishibashi states with continuous U(1)-charge, while the class 1 B-brane only includes the ones with discrete U(1)-charges. Moreover, they have the different boundary wave functions.

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classify consistent D-branes. Application of the method of the bootstrap for disk correlation functions [2, 4] for $\mathcal{N}=2$ Liouville theory will be a difficult but an important task and will be complementary to our analysis.

We have clarified the analogues of the FZZT-branes (extended along the Liouville direction) and the ZZ-branes (localized near the strong coupling region $\phi \sim +\infty$). Among others, the analogues of ZZ-branes play important roles in describing the BPS D-branes wrapped around the vanishing cycles in the Calabi-Yau singularity, and we have shown that the correct intersection numbers are reproduced by the boundary state calculus.

We have also studied briefly the non-BPS D-branes in the 2d type-0 (or type-II) string vacuum constructed from the $\mathcal{N}=2$ Liouville with $\hat{c}=5$. Aspects of unstable branes are quite similar to those of the $\mathcal{N}=1$ model of 2d type-0 string [26, 27]. This is not surprising since they share the same field content as world-sheet theories. However, the $\mathcal{N}=2$ 2d type-0 model may be more challenging, since the *closed* string background is already time-dependent due to the $\mathcal{N}=2$ Liouville potential. The "time-like $\mathcal{N}=2$ Liouville theory" defined by the Wick rotation of Y-direction would be an interesting problem which may shed new light on the study of time-dependent string dynamics. It may be worth pointing out that the time-like $\mathcal{N}=2$ Liouville are free from the subtlety in the analytic continuation of coupling constant in contrast to the bosonic Liouville case (see [73, 74, 75]) where the Liouville field ϕ is regarded as the time direction.

As we have already mentioned, the $\mathcal{N}=2$ Liouville theory with the minimal radius $R=\mathcal{Q}$ is known to be T-dual to the $\mathrm{SL}(2;\mathbb{R})/\,\mathrm{U}(1)$ Kazama-Suzuki model. It is thus an important problem to compare our result with those of the $\mathrm{SL}(2;\mathbb{R})/\,\mathrm{U}(1)$ model. Recently, in [76], the D-branes in the bosonic $\mathrm{SL}(2;\mathbb{R})/\,\mathrm{U}(1)$ model (2d black-hole model) have been studied systematically based on the solutions of boundary bootstrap equations in the $\mathrm{SL}(2;\mathbb{R})$-WZW model given in [77] (see also [78, 79]). Especially, the ZZ-brane analogues localized at the tip of cigar in the 2d black-hole geometry have been constructed. It is natural to expect that supersymmetric version of these states are identified with our class 1 branes in the $\mathcal{N}=2$ Liouville theory by T-duality. Establishing the precise correspondence between $\mathrm{SL}(2;\mathbb{R})/\,\mathrm{U}(1)$ and $\mathcal{N}=2$ Liouville with respect to stable/unstable D-branes is an important open problem.

In this article we have introduced an Ansatz for the allowed set of Ishibashi states based on the considerations of their modular properties. It will be better if one could justify this Ansatz directly from the analysis of the closed string spectrum of the theory. We hope that we can report progress on this issue in a future publication.

Note added in proof: after this paper has been submitted to `hep-th`, a new preprint [80] has appeared which discusses characters of some affine Lie superalgebra and contains results with some overlap with ours. We thank A. Taormina for pointing out this reference.

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A. Notations

A.1 Theta functions

We here summarize our notations of theta functions. We set $q \equiv e^{2\pi i\tau}$ and $y \equiv e^{2\pi i z}$,

$$\begin{aligned}
\theta_1(\tau,z) &= i\sum_{n=-\infty}^{\infty}(-1)^n q^{(n-1/2)^2/2}y^{n-1/2} \\
&\equiv 2\sin(\pi z)q^{1/8}\prod_{m=1}^{\infty}(1-q^m)(1-yq^m)(1-y^{-1}q^m)\,, \\
\theta_2(\tau,z) &= \sum_{n=-\infty}^{\infty}q^{(n-1/2)^2/2}y^{n-1/2} \equiv 2\cos(\pi z)q^{1/8}\prod_{m=1}^{\infty}(1-q^m)(1+yq^m)(1+y^{-1}q^m)\,, \\
\theta_3(\tau,z) &= \sum_{n=-\infty}^{\infty}q^{n^2/2}y^n \equiv \prod_{m=1}^{\infty}(1-q^m)(1+yq^{m-1/2})(1+y^{-1}q^{m-1/2})\,, \\
\theta_4(\tau,z) &= \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2/2}y^n \equiv \prod_{m=1}^{\infty}(1-q^m)(1-yq^{m-1/2})(1-y^{-1}q^{m-1/2})\,, \qquad \text{(A.1)}\\
\Theta_{m,k}(\tau,z) &= \sum_{n=-\infty}^{\infty}q^{k(n+\frac{m}{2k})^2}y^{k(n+\frac{m}{2k})}\,, \qquad \text{(A.2)}\\
\eta(\tau) &= q^{1/24}\prod_{n=1}^{\infty}(1-q^n)\,. \qquad \text{(A.3)}
\end{aligned}$$

A.2 q-gamma function

The "q-gamma function" $S_b(x)$ is defined as [2];

$$\ln S_b(x) = \int_0^\infty \frac{dt}{t}\left\{\frac{\sinh\left(\left(b+\frac{1}{b}-2x\right)t\right)}{2\sinh(bt)\sinh(t/b)} - \frac{b+\frac{1}{b}-2x}{2t}\right\}\,, \qquad \left(\text{for } 0<\mathrm{Re}\;x<b+\frac{1}{b}\right)\,, \tag{A.4}$$

and is analytic continued to other regions of complex x-plane. This function has simple zeros at $x=\frac{m+1}{b}+(n+1)b$ and simple poles at $x=-\frac{m}{b}-nb$ ($m,n\in\mathbb{Z}_{\geq 0}$).

A.3 Character formulas for $\mathcal{N}=2$ minimal model

The easiest way to represent the character formulas of the level k $\mathcal{N}=2$ minimal model ($\hat{c}=k/(k+2)$) is to use its realization as the coset $\mathrm{SU}(2)_k\times\mathrm{U}(1)_2/\mathrm{U}(1)_{k+2}$. We then have the following branching relation;

$$\begin{aligned}
\chi_\ell^{(k)}(\tau,w)\Theta_{s,2}(\tau,w-z) &= \sum_{\substack{m\in\mathbb{Z}_{2(k+2)}\\ \ell+m+s\in 2\mathbb{Z}}}\chi_m^{\ell,s}(\tau,z)\Theta_{m,k+2}\left(\tau,w-\frac{2z}{(k+2)}\right)\,, \\
\chi_m^{\ell,s}(\tau,z) &\equiv 0\,, \qquad \text{for } \ell+m+s\in 2\mathbb{Z}+1\,, \qquad \text{(A.5)}
\end{aligned}$$

where $\chi_\ell^{(k)}(\tau,z)$ is the spin $\ell/2$ character of $\mathrm{SU}(2)_k$;

$$\chi_\ell^{(k)}(\tau,z) = \frac{\Theta_{\ell+1,k+2}(\tau,z) - \Theta_{-\ell-1,k+2}(\tau,z)}{\Theta_{1,2}(\tau,z) - \Theta_{-1,2}(\tau,z)}\,. \tag{A.6}$$

Then, the desired character formulas are written as

$$\mathrm{ch}^{(\mathrm{NS})}_{\ell,m}(\tau,z) = \chi_m^{\ell,0}(\tau,z) + \chi_m^{\ell,2}(\tau,z)\,, \tag{A.7}$$

$$\mathrm{ch}^{(\widetilde{\mathrm{NS}})}_{\ell,m}(\tau,z) = \chi_m^{\ell,0}(\tau,z) - \chi_m^{\ell,2}(\tau,z) \equiv e^{-i\pi\frac{m}{k+2}}\mathrm{ch}^{(\mathrm{NS})}_{\ell,m}\left(\tau, z+\frac{1}{2}\right), \tag{A.8}$$

$$\mathrm{ch}^{(\mathrm{R})}_{\ell,m}(\tau,z) = \chi_m^{\ell,1}(\tau,z) + \chi_m^{\ell,3}(\tau,z) \equiv q^{\frac{k}{8(k+2)}}y^{\frac{k}{2(k+2)}}\mathrm{ch}^{(\mathrm{NS})}_{\ell,m+1}\left(\tau, z+\frac{\tau}{2}\right), \tag{A.9}$$

$$\begin{aligned}\mathrm{ch}^{(\widetilde{\mathrm{R}})}_{\ell,m}(\tau,z) &= \chi_m^{\ell,1}(\tau,z) - \chi_m^{\ell,3}(\tau,z) \\ &\equiv -e^{-i\pi\frac{m+1}{k+2}}q^{\frac{k}{8(k+2)}}y^{\frac{k}{2(k+2)}}\mathrm{ch}^{(\mathrm{NS})}_{\ell,m+1}\left(\tau, z+\frac{1}{2}+\frac{\tau}{2}\right).\end{aligned} \tag{A.10}$$

By definition, we may restrict to $\ell+m \in 2\mathbb{Z}$ for the NS and $\widetilde{\mathrm{NS}}$ sectors, and to $\ell+m \in 2\mathbb{Z}+1$ for the R and $\widetilde{\mathrm{R}}$ sectors. The modular transformation coefficients (4.7) can be directly read off from these formulas.

B. A useful formula for modular calculus

We here present a useful formula relevant for the calculation of S-transformation of the massless characters, which has been proved in [48]. Define the following function

$$I(k,a,b;\tau,z) \equiv \sum_{r\in\mathbb{Z}+\frac{1}{2}} e^{2\pi i a r}\,\frac{(yq^r)^b}{1+yq^r}\,y^{kr}q^{\frac{k}{2}r^2}\,, \qquad (a,b\in\mathbb{R},\ k>0)\,, \tag{B.1}$$

then we can prove the following identity[12]

$$\begin{aligned}\frac{i}{\tau}e^{-ik\frac{\pi z^2}{\tau}}\,I\left(k,a,b;-\frac{1}{\tau},\frac{z}{\tau}\right) &= \sum_{r\in\mathbb{Z}+a} e^{i\pi(r-a)}y^r q^{\frac{r^2}{2k}}\,\frac{1}{\sqrt{k}}\int_{-\infty}^{\infty}dp\,\frac{e^{-2\pi b\left(\frac{p}{\sqrt{k}}+i\frac{r}{k}\right)}}{1+e^{-2\pi\left(\frac{p}{\sqrt{k}}+i\frac{r}{k}\right)}}\,q^{\frac{p^2}{2}} + \\ &\quad + i\sum_{\substack{s\in\mathbb{Z}+\frac{1}{2}\\ \delta(a,s)\neq 0}} e^{i\pi(\delta(a,s)-a)}e^{2\pi i\left(\frac{k}{2}-b\right)s}\,\frac{(yq^s)^{\delta(a,s)}}{1+yq^s}\,y^{ks}q^{\frac{k}{2}s^2} + \\ &\quad + \frac{i}{2}\sum_{\substack{s\in\mathbb{Z}+\frac{1}{2}\\ \delta(a,s)=0}} e^{-i\pi a}e^{2\pi i\left(\frac{k}{2}-b\right)s}\,\frac{1-yq^s}{1+yq^s}\,y^{ks}q^{\frac{k}{2}s^2}\,,\end{aligned} \tag{B.2}$$

where $\delta(a,s)$ is a real number uniquely determined by the conditions;

$$\begin{aligned}\delta(a,s) &\equiv a-ks \pmod{\mathbb{Z}}\,,\\ 0 &\leq \delta(a,s) < 1\,.\end{aligned} \tag{B.3}$$

(We have slightly modified the original notations in [48] for our convenience.)

[12] If the integrand in the r.h.s. of (B.2) has a pole on the real axis, we must regard the integral as the principal value.

C. Modular transformation formulas of characters of the extended chiral algebras

We here summarize the modular transformation formulas of characters of the extended chiral algebra defined by adding the spectral flow operators to the $\mathcal{N}=2$ algebra in the cases $\hat{c}=2,3,4,5$. The most familiar example is $\hat{c}=2$ case. The extended algebra is nothing but the $\mathcal{N}=4$ SCA of level 1 and is relevant for the $K3$ compactification [44, 45]. Generalizations to $\hat{c}=3$ (CY_3 compactification) and $\hat{c}=4$ (CY_4 compactification) are discussed in [46, 47].

The desired characters are obtained by summing up the irreducible characters of $\mathcal{N}=2$ SCA (2.3), (2.4), (2.5) over the integral spectral flows.

$$
\begin{aligned}
\mathrm{Ch}^{(\mathrm{NS})}(h,Q;\tau,z) &= \sum_{m\in\mathbb{Z}} q^{\frac{\hat{c}}{2}m^2} y^{\hat{c}m} \mathrm{ch}^{(\mathrm{NS})}(h,Q;\tau,z+m\tau) \\
&= q^{h-\frac{Q^2}{2(\hat{c}-1)}-\frac{\hat{c}-1}{8}} \Theta_{Q,\frac{\hat{c}-1}{2}}(\tau,2z)\, \frac{\theta_3(\tau,z)}{\eta(\tau)^3}\,, \qquad (\mathrm{C.1}) \\
\mathrm{Ch}^{(\mathrm{NS})}_{\mathbf{M}}(Q(\neq 0);\tau,z) &= \sum_{m\in\mathbb{Z}} q^{\frac{\hat{c}}{2}m^2} y^{\hat{c}m} \mathrm{ch}^{(\mathrm{NS})}_{\mathbf{M}}(Q;\tau,z+m\tau) \\
&= q^{-\frac{\hat{c}-1}{8}} \sum_{m\in\mathbb{Z}} \frac{q^{\frac{\hat{c}-1}{2}m^2+|Q|m+\frac{|Q|}{2}} y^{\mathrm{sgn}(Q)(|Q|+(\hat{c}-1)m)}}{1+y^{\mathrm{sgn}(Q)}q^{m+1/2}}\, \frac{\theta_3(\tau,z)}{\eta(\tau)^3}\,, \qquad (\mathrm{C.2}) \\
\mathrm{Ch}^{(\mathrm{NS})}_{\mathbf{G}}(\tau,z) &= \sum_{m\in\mathbb{Z}} q^{\frac{\hat{c}}{2}m^2} y^{\hat{c}m} \mathrm{ch}^{(\mathrm{NS})}_{\mathbf{G}}(\tau,z+m\tau) \\
&= q^{-\frac{\hat{c}-1}{8}} \sum_{m\in\mathbb{Z}} \frac{(1-q)q^{\frac{\hat{c}-1}{2}m^2+m-\frac{1}{2}} y^{(\hat{c}-1)m+1}}{(1+yq^{m+1/2})(1+yq^{m-1/2})}\, \frac{\theta_3(\tau,z)}{\eta(\tau)^3}\,. \qquad (\mathrm{C.3})
\end{aligned}
$$

Note that, in the cases of $\hat{c}=3,5$ these characters themselves are identified as the extended characters $\chi_*(*;\tau,z)$ defined in section 2, while in the other cases $\hat{c}=2,4$ one takes the sum $\chi_*(r=0,*;\tau,z)+\chi_*(r=-1,*;\tau,z)$ to define the characters of the extended algebra. Furthermore, the sectors with odd quantum numbers j, s are eliminated by the locality with the integral spectral flow generators $U_{\pm 1}$, rather than $U_{\pm 2}$.

The extended characters for other spin structures are defined by the 1/2-spectral flow;[13]

$$
\begin{aligned}
\mathrm{Ch}^{(\widetilde{\mathrm{NS}})}_*(*;\tau,z) &\equiv \mathrm{Ch}^{(\mathrm{NS})}_*(*;\tau,z+\frac{1}{2})\,, \\
\mathrm{Ch}^{(\mathrm{R})}_*(*;\tau,z) &\equiv q^{\hat{c}/8}y^{\hat{c}/2}\mathrm{Ch}^{(\mathrm{NS})}_*(*;\tau,z+\frac{\tau}{2})\,, \\
\mathrm{Ch}^{(\widetilde{\mathrm{R}})}_*(*;\tau,z) &\equiv q^{\hat{c}/8}y^{\hat{c}/2}\mathrm{Ch}^{(\mathrm{NS})}_*(*;\tau,z+\frac{\tau}{2}+\frac{1}{2})\,. \qquad (\mathrm{C.4})
\end{aligned}
$$

We here used the abbreviated notations which should be understood in the meaning as in (C.1), (C.2) or (C.3).

[13] We shall here adopt the convention such that the quantum numbers appearing in the Ramond characters are the same as the corresponding NS ones. Namely, (h,Q) in the Ramond characters are *not* equal the conformal weights and U(1)-charges of the Ramond vacuum states. Although this convention may be somewhat confusing, it is convenient to write down the modular transformation formulas for general spin structures.

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First of all, it is quite easy to evaluate the T-transformation. We obtain

$$\begin{aligned} \mathrm{Ch}^{(\sigma)}(h,Q;\tau+1,z) &= e^{2\pi i\gamma(h,\sigma)}\,\mathrm{Ch}^{(T\cdot\sigma)}(h,Q;\tau,z)\,, \\ \mathrm{Ch}^{(\sigma)}_{\mathbf{M}}(Q;\tau+1,z) &= e^{2\pi i\gamma(|Q|/2,\sigma)}\,\mathrm{Ch}^{(T\cdot\sigma)}_{\mathbf{M}}(Q;\tau,z)\,, \\ \mathrm{Ch}^{(\sigma)}_{\mathbf{G}}(\tau+1,z) &= e^{2\pi i\gamma(0,\sigma)}\,\mathrm{Ch}^{(T\cdot\sigma)}_{\mathbf{G}}(\tau,z)\,, \end{aligned} \tag{C.5}$$

where we set

$$\gamma(h,\sigma)\equiv\begin{cases} h-\frac{\hat{c}}{8} & (\sigma=\mathrm{NS},\,\widetilde{\mathrm{NS}}) \\ h & (\sigma=\mathrm{R},\,\widetilde{\mathrm{R}}) \end{cases} \tag{C.6}$$

and $T\cdot\mathrm{NS}=\widetilde{\mathrm{NS}}$, $T\cdot\widetilde{\mathrm{NS}}=\mathrm{NS}$, $T\cdot\mathrm{R}=\mathrm{R}$, $T\cdot\widetilde{\mathrm{R}}=\widetilde{\mathrm{R}}$.

Let us next consider the S-transformation. For the massive representations, we have the following continuous spectra. The unitarity condition is $h>|Q|/2$ in all the cases.

- $\hat{c}=2$: We have only one continuous series generated by the highest-weights state $(h,Q=0)$.
- $\hat{c}=3$: We have two continuous series $(h,Q=0)$, $(h,Q=\pm1)$. (the vacua are doubly degenerated in the second case).
- $\hat{c}=4$: We have three continuous series $(h,Q=0)$, $(h,Q=+1)$ and $(h,Q=-1)$.
- $\hat{c}=5$: We have four continuous series $(h,Q=0)$, $(h,Q=+1)$, $(h,Q=-1)$ and $(h,Q=\pm2)$ (the vacua are doubly degenerated in the last case).

Since the massive character is written as in (C.1), the S-transformation formula is easily follows from that of the theta function;

$$\begin{aligned} &\mathrm{Ch}^{(\sigma)}\left(h=\frac{p^2}{2}+\frac{Q^2}{2(\hat{c}-1)}+\frac{\hat{c}-1}{8},Q;-\frac{1}{\tau},\frac{z}{\tau}\right)= \\ &=\kappa(\sigma)e^{i\pi\frac{\hat{c}z^2}{\tau}}\frac{2}{\sqrt{\hat{c}-1}}\int_0^\infty dp'\cos(2\pi pp')\times \\ &\times\sum_{Q'\in\mathbb{Z}_{\hat{c}-1}}e^{-2\pi i\frac{QQ'}{\hat{c}-1}}\mathrm{Ch}^{(S\cdot\sigma)}\left(h=\frac{p'^2}{2}+\frac{Q'^2}{2(\hat{c}-1)}+\frac{\hat{c}-1}{8},Q';\tau,z\right), \end{aligned} \tag{C.7}$$

where we set

$$\kappa(\sigma)=\begin{cases} 1 & \sigma=\mathrm{NS},\,\widetilde{\mathrm{NS}},\,\mathrm{R} \\ e^{-i\pi\frac{\hat{c}}{2}} & \sigma=\widetilde{\mathrm{R}} \end{cases}, \tag{C.8}$$

and $S\cdot\mathrm{NS}=\mathrm{NS}$, $S\cdot\widetilde{\mathrm{NS}}=\mathrm{R}$, $S\cdot\mathrm{R}=\widetilde{\mathrm{NS}}$, $S\cdot\widetilde{\mathrm{R}}=\widetilde{\mathrm{R}}$.

The formulas for the massless characters are much more non-trivial. We can evaluate them by using the formula (B.2). They are summarized as follows;

1. $\hat{c}=2$: we have two massless representations $Q=0$ (graviton representation) and $Q=\pm 1$ (doubly degenerated vacua). The modular transformation formulas are written as [44];

$$\begin{aligned}
\mathrm{Ch}^{(\sigma)}_{\mathbf{G}}\left(-\frac{1}{\tau},\frac{z}{\tau}\right) &= \kappa(\sigma)e^{i\pi\frac{2z^2}{\tau}}\times \\
&\quad\times\Big\{2\mathrm{Ch}^{(S\cdot\sigma)}_{\mathbf{M}}(|Q|=1;\tau,z)+ \\
&\quad+2\int_0^\infty dp'\,\sinh(\pi p')\tanh(\pi p')\,\mathrm{Ch}^{(S\cdot\sigma)}\times \\
&\quad\times\left(h=\frac{p'^2}{2}+\frac{1}{8},Q=0;\tau,z\right)\Big\}, \qquad \text{(C.9)}\\
\mathrm{Ch}^{(\sigma)}_{\mathbf{M}}\left(|Q|=1;-\frac{1}{\tau},\frac{z}{\tau}\right) &= \kappa(\sigma)e^{i\pi\frac{2z^2}{\tau}}\times \\
&\quad\times\Big\{-\mathrm{Ch}^{(S\cdot\sigma)}_{\mathbf{M}}(|Q|=1;\tau,z)+\int_0^\infty dp'\,\frac{1}{\cosh(\pi p')}\times \\
&\quad\times\mathrm{Ch}^{(S\cdot\sigma)}\left(h=\frac{p'^2}{2}+\frac{1}{8},Q=0;\tau,z\right)\Big\}. \qquad \text{(C.10)}
\end{aligned}$$

2. $\hat{c}=3$: we have three massless representations $Q=0$, $Q=+1$ and $Q=-1$. The modular transformation formulas are written as [46];

$$\begin{aligned}
\mathrm{Ch}^{(\sigma)}_{\mathbf{G}}\left(-\frac{1}{\tau},\frac{z}{\tau}\right) &= \kappa(\sigma)e^{i\pi\frac{3z^2}{\tau}}\sqrt{2}\int_0^\infty dp'\,\sinh\left(\sqrt{2}\pi p'\right)\times \\
&\quad\times\Big\{\tanh\left(\frac{\pi p'}{\sqrt{2}}\right)\mathrm{Ch}^{(S\cdot\sigma)}\left(h=\frac{p'^2}{2}+\frac{1}{4},Q=0;\tau,z\right)+ \\
&\quad+\coth\left(\frac{\pi p'}{\sqrt{2}}\right)\mathrm{Ch}^{(S\cdot\sigma)}\left(h=\frac{p'^2}{2}+\frac{1}{2},|Q|=1;\tau,z\right)\Big\}, \\
\mathrm{Ch}^{(\sigma)}_{\mathbf{M}}\left(Q=\pm1;-\frac{1}{\tau},\frac{z}{\tau}\right) &= \kappa(\sigma)e^{i\pi\frac{3z^2}{\tau}}\times \\
&\quad\times\Big[\frac{1}{\sqrt{2}}\int_0^\infty dp'\Big\{\mathrm{Ch}^{(S\cdot\sigma)}\left(h=\frac{p'^2}{2}+\frac{1}{4},Q=0;\tau,z\right)- \\
&\quad-\mathrm{Ch}^{(S\cdot\sigma)}\left(h=\frac{p'^2}{2}+\frac{1}{2},|Q|=1;\tau,z\right)\Big\} \\
&\quad-\frac{i}{2}\Big\{\mathrm{Ch}^{(S\cdot\sigma)}_{\mathbf{M}}(Q=\pm1;\tau,z)- \\
&\quad-\mathrm{Ch}^{(S\cdot\sigma)}_{\mathbf{M}}(Q=\mp1;\tau,z)\Big\}\Big]. \qquad \text{(C.11)}
\end{aligned}$$

3. $\hat{c}=4$: we have four massless representations $Q=0$, $Q=+1$, $Q=-1$ and $Q=\pm2$ (doubly degenerated vacua). The modular transformation formulas are written as;

$$\mathrm{Ch}^{(\sigma)}_{\mathbf{G}}\left(-\frac{1}{\tau},\frac{z}{\tau}\right)=$$

$$
\begin{aligned}
&= e^{i\pi\frac{4z^2}{\tau}}\Bigg[2\mathrm{Ch}^{(S\cdot\sigma)}_{\mathbf{M}}(|Q|=2;\tau,z) + \frac{2}{\sqrt{3}}\int_0^\infty dp'\ \sinh(\sqrt{3}\pi p')\ \times \\
&\qquad \times \tanh\left(\frac{\pi p'}{\sqrt{3}}\right)\mathrm{Ch}^{(S\cdot\sigma)}\left(h=\frac{p'^2}{2}+\frac{3}{8}, Q=0;\tau,z\right) + \\
&\qquad + \frac{1}{\sqrt{3}}\int_0^\infty dp'\ \frac{\sinh(\sqrt{3}\pi p')\sinh\left(\frac{2\pi p'}{\sqrt{3}}\right)}{\left|\cosh\left(\frac{\pi p'}{\sqrt{3}}+\frac{i\pi}{3}\right)\right|^2}\times \\
&\qquad \times \left\{\mathrm{Ch}^{(S\cdot\sigma)}\left(h=\frac{p'^2}{2}+\frac{13}{24}, Q=1;\tau,z\right) + \right.\\
&\qquad\quad \left.\left. + \mathrm{Ch}^{(S\cdot\sigma)}\left(h=\frac{p'^2}{2}+\frac{13}{24}, Q=-1;\tau,z\right)\right\}\right], \qquad \text{(C.12)}
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{Ch}^{(\sigma)}_{\mathbf{M}}\left(Q=\pm1;-\frac{1}{\tau},\frac{z}{\tau}\right) = \\
&= e^{i\pi\frac{4z^2}{\tau}}\Bigg[-\mathrm{Ch}^{(S\cdot\sigma)}_{\mathbf{M}}(|Q|=2;\tau,z) + \frac{1}{2\sqrt{3}}\int_0^\infty dp'\ \times \\
&\qquad \times \left\{\frac{2\cosh\left(\frac{2\pi p'}{\sqrt{3}}\right)}{\cosh\left(\frac{\pi p'}{\sqrt{3}}\right)}\mathrm{Ch}^{(S\cdot\sigma)}\left(h=\frac{p'^2}{2}+\frac{3}{8}, Q=0;\tau,z\right) - \right.\\
&\qquad\quad - \frac{e^{i\frac{\pi}{3}}\cosh(\sqrt{3}\pi p') - \cosh\left(\frac{\pi p'}{\sqrt{3}}\right)}{\left|\cosh\left(\frac{\pi p'}{\sqrt{3}}+i\frac{\pi}{3}\right)\right|^2}\times \\
&\qquad\quad \times \mathrm{Ch}^{(S\cdot\sigma)}\left(h=\frac{p'^2}{2}+\frac{13}{24}, Q=\pm1;\tau,z\right) - \\
&\qquad\quad - \frac{e^{-i\frac{\pi}{3}}\cosh(\sqrt{3}\pi p') - \cosh\left(\frac{\pi p'}{\sqrt{3}}\right)}{\left|\cosh\left(\frac{\pi p'}{\sqrt{3}}+i\frac{\pi}{3}\right)\right|^2}\times \\
&\qquad\quad \left.\left. \times \mathrm{Ch}^{(S\cdot\sigma)}\left(h=\frac{p'^2}{2}+\frac{13}{24}, Q=\mp1;\tau,z\right)\right\}\right], \qquad \text{(C.13)}
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{Ch}^{(\sigma)}_{\mathbf{M}}\left(|Q|=2;-\frac{1}{\tau},\frac{z}{\tau}\right) = \\
&= e^{i\pi\frac{4z^2}{\tau}}\Bigg[\mathrm{Ch}^{(S\cdot\sigma)}_{\mathbf{M}}(|Q|=2;\tau,z) + \frac{1}{2\sqrt{3}}\int_0^\infty dp'\ \times \\
&\qquad \times \left\{\frac{2}{\cosh\left(\frac{\pi p'}{\sqrt{3}}\right)}\mathrm{Ch}^{(S\cdot\sigma)}\left(h=\frac{p'^2}{2}+\frac{3}{8}, Q=0;\tau,z\right) - \right.\\
&\qquad\quad - \frac{\cosh\left(\frac{\pi p'}{\sqrt{3}}\right)}{\left|\cosh\left(\frac{\pi p'}{\sqrt{3}}+i\frac{\pi}{3}\right)\right|^2}\times
\end{aligned}
$$

$$\times\left(\mathrm{Ch}^{(S\cdot\sigma)}\left(h=\frac{p'^2}{2}+\frac{13}{24},Q=1;\tau,z\right)+\right.$$
$$\left.\left.\left.+\,\mathrm{Ch}^{(S\cdot\sigma)}\left(h=\frac{p'^2}{2}+\frac{13}{24},Q=-1;\tau,z\right)\right)\right\}\right]. \quad \text{(C.14)}$$

4. $\hat{c}=5$**:** we have five massless representations $Q=0$, $Q=+1$, $Q=-1$, $Q=+2$ and $Q=-2$. The S-transformation formulas are

$$\mathrm{Ch}^{(\sigma)}_{\mathbf{G}}\left(-\frac{1}{\tau},\frac{z}{\tau}\right)=$$
$$=\kappa(\sigma)e^{i\pi\frac{5z^2}{\tau}}\int_0^\infty dp'\sinh(2\pi p')\times$$
$$\times\left[\tanh\left(\frac{\pi p'}{2}\right)\mathrm{Ch}^{(S\cdot\sigma)}\left(h=\frac{p'^2}{2}+\frac{1}{2},Q=0;\tau,z\right)\right.$$
$$+\tanh(\pi p')\left\{\mathrm{Ch}^{(S\cdot\sigma)}\left(h=\frac{p'^2}{2}+\frac{5}{8},Q=1;\tau,z\right)+\right.$$
$$\left.+\,\mathrm{Ch}^{(S\cdot\sigma)}\left(h=\frac{p'^2}{2}+\frac{5}{8},Q=-1;\tau,z\right)\right\}+$$
$$\left.+\coth\left(\frac{\pi p'}{2}\right)\mathrm{Ch}^{(S\cdot\sigma)}\left(h=\frac{p'^2}{2}+1,|Q|=2;\tau,z\right)\right], \quad \text{(C.15)}$$

$$\mathrm{Ch}^{(\sigma)}_{\mathbf{M}}\left(Q=\pm1;-\frac{1}{\tau},\frac{z}{\tau}\right)=$$
$$=\kappa(\sigma)e^{i\pi\frac{5z^2}{\tau}}\left[\frac{1}{2}\int_0^\infty dp'\times\right.$$
$$\times\left\{\frac{\cosh\left(\frac{3\pi p'}{2}\right)}{\cosh\left(\frac{\pi p'}{2}\right)}\mathrm{Ch}^{(S\cdot\sigma)}\left(\frac{p'^2}{2}+\frac{1}{2},0;\tau,z\right)+\right.$$
$$+\left(1-\frac{i\cosh(2\pi p')}{\cosh(\pi p')}\right)\mathrm{Ch}^{(S\cdot\sigma)}\left(\frac{p'^2}{2}+\frac{5}{8},\pm1;\tau,z\right)+$$
$$+\left(1+\frac{i\cosh(2\pi p')}{\cosh(\pi p')}\right)\mathrm{Ch}^{(S\cdot\sigma)}\left(\frac{p'^2}{2}+\frac{5}{8},\mp1;\tau,z\right)-$$
$$\left.-\frac{\sinh\left(\frac{3\pi p'}{2}\right)}{\sinh\left(\frac{\pi p'}{2}\right)}\mathrm{Ch}^{(S\cdot\sigma)}\left(\frac{p'^2}{2}+\frac{1}{2},2;\tau,z\right)\right\}-$$
$$\left.-\frac{i}{2}\left\{\mathrm{Ch}^{(S\cdot\sigma)}_{\mathbf{M}}(Q=\pm2;\tau,z)-\mathrm{Ch}^{(S\cdot\sigma)}_{\mathbf{M}}(Q=\mp2;\tau,z)\right\}\right], \quad \text{(C.16)}$$

$$\mathrm{Ch}^{(\sigma)}_{\mathbf{M}}\left(Q=\pm2;-\frac{1}{\tau},\frac{z}{\tau}\right)=$$
$$=\kappa(\sigma)e^{i\pi\frac{5z^2}{\tau}}\left[\frac{1}{2}\int_0^\infty dp'\times\right.$$

$$
\begin{aligned}
&\times\left\{\mathrm{Ch}^{(S\cdot\sigma)}\left(\frac{p'^2}{2}+\frac{1}{2},0;\tau,z\right)-\right.\\
&\qquad-\left(1+\frac{i}{\cosh(\pi p')}\right)\mathrm{Ch}^{(S\cdot\sigma)}\left(\frac{p'^2}{2}+\frac{5}{8},\pm1;\tau,z\right)-\\
&\qquad-\left(1-\frac{i}{\cosh(\pi p')}\right)\mathrm{Ch}^{(S\cdot\sigma)}\left(\frac{p'^2}{2}+\frac{5}{8},\mp1;\tau,z\right)+\\
&\qquad\left.+\,\mathrm{Ch}^{(S\cdot\sigma)}\left(\frac{p'^2}{2}+\frac{1}{2},2;\tau,z\right)\right\} \qquad\qquad (C.17)\\
&\left.+\frac{i}{2}\left\{\mathrm{Ch}^{(S\cdot\sigma)}_{\mathbf{M}}(Q=\pm2;\tau,z)-\mathrm{Ch}^{(S\cdot\sigma)}_{\mathbf{M}}(Q=\mp2;\tau,z)\right\}\right],
\end{aligned}
$$

D. Analysis on the class 3 states: concrete examples

Checking the Cardy condition for the overlaps among the class 3 states is a non-trivial problem. Here we try to do this task for the special cases $\hat{c}=2,3,4,5$ (values of (N,K) are given by $(2,1),(1,1),(2,3),(1,2)$, respectively).

Since we are interested in the BPS D-branes in the type-II string vacua: $\mathbb{R}^{d-1,1}\times\mathcal{N}=2$ Liouville ($\hat{c}+d/2=5$), we shall only treat the sectors with integral U(1)-charges in the closed as well as open string channel. This restriction amounts to only considering the boundary states preserved by the extended chiral algebras, and thus the relevant conformal blocks should be expanded by the characters of extended chiral algebras $\mathrm{Ch}^{(*)}(h,Q;\tau,z)$, $\mathrm{Ch}^{(*)}_{\mathbf{M}}(Q;\tau,z)$ and $\mathrm{Ch}^{(*)}_{\mathbf{G}}(\tau,z)$. Their modular transformation formulas are quite useful for the calculation of cylinder amplitudes and summarized in appendix C. For simplicity, we concentrate on the NSNS part of the boundary states and omit the insertion of the GSO projection $1+(-1)^F/2$, which amount to only considering the NS sector in the open string channel. One can easily obtain the results for other spin structures by spectral flow.

1. $\hat{c}=2$**:** the $\hat{c}=2$ case corresponds to $N=2$, $K=1$ ($\mathcal{Q}=1$). The class 1 and class 2 states are written as

$$
\begin{aligned}
&\textbf{class 1 states:}\quad |B;r=0\rangle+|B;r=-1\rangle\ ,\\
&\textbf{class 2 states:}\quad |B;p,j=0\rangle+|B;p,j=2\rangle\quad\left(\frac{p^2}{2}>-\frac{1}{32}\right). \qquad (D.1)
\end{aligned}
$$

We have taken account of the projection to integral U(1)-charges as in section 4. It is obvious that $|B;O\rangle\equiv|B;r=0\rangle$. We have also imposed the condition (3.57) for the class 2 states. Note that $|B;p,j=\pm1\rangle$ correspond to the massive representation with half-integral U(1)-charges and thus we exclude them.

Since we are imposing the integrality of U(1)-charge, we just have a unique possibility of the class 3 states $[(0,2),\,(-1,2)]$ which corresponds to the combination of the characters as

$$
\chi_{\mathbf{M}}(r=0,s=2;\tau,z)+\chi_{\mathbf{M}}(r=-1,s=2;\tau,z)\equiv\mathrm{Ch}^{(\mathrm{NS})}_{\mathbf{M}}(|Q|=1;\tau,z)\,. \qquad (D.2)
$$

We thus denote this state as $|B;|Q|=1\rangle$. It is easy to calculate the overlap with itself, and we find that it cannot be written in the form (3.4) with any appropriate spectral density. We thus conclude that Cardy states are given by the set (D.1) and no class 3 states are allowed.

2. $\hat{c}=3$: $\hat{c}=3$ corresponds to $N=K=1$ ($\mathcal{Q}=\sqrt{2}$). The class 1 and class 2 states are given as

$$\begin{aligned}
&\textbf{class 1 states:} \quad |B;r=0\rangle \equiv |B;O\rangle \ , \\
&\textbf{class 2 states:} \quad |B;p,j=0\rangle \quad \left(\frac{p^2}{2} > -\frac{1}{16}\right), \quad |B;p,j=1\rangle \quad \left(\frac{p^2}{2} \geq 0\right). \qquad \text{(D.3)}
\end{aligned}$$

We only have a unique possibility of the class 3 state; $[(0,1),\ (0,2)]$. However, because the following character identity holds;

$$\begin{aligned}
&\chi_{\mathbf{M}}(0,1;\tau,z) + \chi_{\mathbf{M}}(0,2;\tau,z) = \\
&\quad = \chi(p=0,j=1;\tau,z)\left(\equiv \mathrm{Ch}^{(\mathrm{NS})}\left(h=\frac{1}{2}, |Q|=1;\tau,z\right)\right), \qquad \text{(D.4)}
\end{aligned}$$

it is a special case of the class 2 states (D.3). We thus conclude that only (D.3) are allowed as the Cardy states.

3. $\hat{c}=4$: $\hat{c}=4$ corresponds to $N=2$, $K=3$ ($\mathcal{Q}=\sqrt{3}$). This is the most non-trivial example. The class 1 and class 2 state are given as

$$\begin{aligned}
&\textbf{class 1 states:} \quad |B;r=0\rangle (\equiv |B;O\rangle) + |B;r=-1\rangle \\
&\textbf{class 2 states:} \quad |B;p,2j\rangle + |B;p,2j+6\rangle \qquad \text{(D.5)} \\
&\qquad \left(j=0,1,2\ ,\ \ \frac{p^2}{2} > -\frac{3}{32}\ \ (j=0)\ ,\ \ \frac{p^2}{2} \geq -\frac{1}{24}\ \ (j=1,2)\right).
\end{aligned}$$

We again have taken account of the projection to the closed string sectors with integral U(1)-charges. Note that $|B;p,2j+1\rangle + |B;p,2j+7\rangle$ are eliminated by our assumption for the Cardy states. All the class 2 states above are compatible with unitarity even in the cases of imaginary p.

We have many possibilities of the class 3 states. Fortunately, many of the pairs defining the class 3 states (3.37) are eliminated by the assumption of charge integrality, or reduced to the class 2 states by the character identities similar to (D.4). Only three candidates remain: $[(0,2),\ (-1,2)]$, $[(0,4),\ (-1,4)]$, $[(0,6),\ (-1,6)]$, which respectively correspond to the massless characters;

$$\begin{aligned}
\chi_{\mathbf{M}}(0,2;\tau,z) + \chi_{\mathbf{M}}(-1,2;\tau,z) &= \mathrm{Ch}^{(\mathrm{NS})}_{\mathbf{M}}(Q=1;\tau,z)\,, \\
\chi_{\mathbf{M}}(0,4;\tau,z) + \chi_{\mathbf{M}}(-1,4;\tau,z) &= \mathrm{Ch}^{(\mathrm{NS})}_{\mathbf{M}}(|Q|=2;\tau,z)\,, \\
\chi_{\mathbf{M}}(0,6;\tau,z) + \chi_{\mathbf{M}}(-1,6;\tau,z) &= \mathrm{Ch}^{(\mathrm{NS})}_{\mathbf{M}}(Q=-1;\tau,z)\,, \qquad \text{(D.6)}
\end{aligned}$$

and hence we denote them as $|B;Q=1\rangle$, $|B;Q=-1\rangle$ and $|B;|Q|=2\rangle$.

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However, all of them are found not to yield positive spectral densities in their overlap amplitudes. We thus look for the proper Cardy states satisfying (3.4) among the linear combinations

$$|B;(m_+,m_-,n)\rangle\!\rangle = m_+|B;Q{=}1\rangle + m_-|B;Q{=}-1\rangle + n|B;|Q|{=}2\rangle\,, \quad m_+,m_-,n \in \mathbb{Z}_{\geq 0}\,. \tag{D.7}$$

We set

$$\begin{aligned} \mathbf{S}_+ &= \{|B;(m_+,m_-,n)\rangle\!\rangle \mid -m_+ - m_- + n > 0\}\,, \\ \mathbf{S}_- &= \{|B;(m_+,m_-,n)\rangle\!\rangle \mid -m_+ - m_- + n < 0\}\,, \end{aligned} \tag{D.8}$$

where $m_+, m_-, n \in \mathbb{Z}_{\geq 0}$ and we assume at least two of integers m_+, m_-, n are non-zero. Then, the careful analysis on overlaps leads to the results;

- The overlaps among any pair of states (D.7) both belonging to the same set $\mathbf{S}_+$ (or $\mathbf{S}_-$) satisfy the Cardy condition (3.4) with the positive spectral densities.
- Any pairing between the states of $\mathbf{S}_+$ and $\mathbf{S}_-$ cannot yield the positive spectral densities.

For example, we obtain after a lengthy calculations

$$\begin{aligned} & e^{\pi\frac{4z^2}{T}}\langle B;(m_+,m_-,0)|e^{-\pi T H^{(c)}} e^{i\pi z(J_0+\tilde{J}_0)}|B;(m'_+,m'_-,0)\rangle = \\ & \quad = \int_0^\infty dp \sum_{Q=0,\pm 1} \rho_Q(p|m_+,m_-,m'_+,m'_-)\mathrm{Ch}^{(\mathrm{NS})}\left(h=\frac{p^2}{2}+\frac{Q^2+9}{24}, Q; it, z'\right) + \\ & \qquad + (m_+ + m_-)(m'_+ + m'_-)\mathrm{Ch}^{(\mathrm{NS})}_{\mathbf{M}}(|Q|=2; it, z')\,, \end{aligned} \tag{D.9}$$

$$\begin{aligned} & \rho_0(p|m_+,m_-,m'_+,m'_-) = \\ & \quad = 2(m_+m'_+ + m_-m'_-)\int_0^\infty dp'\,\frac{\cos(2\pi pp')\cosh(\pi\sqrt{3}p')}{\sinh(\pi\sqrt{3}p')\sinh\left(\frac{\pi p'}{\sqrt{3}}\right)}\,, \end{aligned} \tag{D.10}$$

$$\begin{aligned} & \rho_{\pm 1}(p|m_+,m_-,m'_+,m'_-) = \\ & \quad = \int_0^\infty dp'\,\frac{\cos(2\pi pp')}{\sinh(\pi\sqrt{3}p')\sinh\left(\frac{\pi p'}{\sqrt{3}}\right)} \times \\ & \qquad \times \left\{ \left((m_+ + m_-)(m'_+ + m'_-) - 2m_\pm m'_\mp + 2m_\mp m'_\pm\right)\cosh\left(\frac{\pi p'}{\sqrt{3}}\right) - \right. \\ & \qquad\quad \left. - \left((m_+ + m_-)(m'_+ + m'_-) - 2m_\pm m'_\mp\right)\cosh\left(\pi\sqrt{3}p'\right) \right\}, \end{aligned} \tag{D.11}$$

where the characters of the $\hat{c}=4$ extended algebra are given as

$$\mathrm{Ch}^{(\mathrm{NS})}\left(\frac{p^2}{2}+\frac{Q^2+9}{24}, Q; \tau, z\right) = \chi(p, 2Q; \tau, z) + \chi(p, 2Q+6; \tau, z)\,, \tag{D.12}$$

and also by (D.6). The spectral densities (D.10) and (D.11) are defined with the IR regularization considered before. They are positive for arbitrary $m_+, m_-, m'_+, m'_- \in \mathbb{Z}_{>0}$, and rewritten by using the q-gamma function (A.4).

We here make a comment: we have a character identity

$$\begin{aligned}\mathrm{Ch}^{(\mathrm{NS})}_{\mathbf{M}}(Q{=}\pm1;\tau,z)+\mathrm{Ch}^{(\mathrm{NS})}_{\mathbf{M}}(|Q|{=}2;\tau,z) \;&=\; \mathrm{Ch}^{(\mathrm{NS})}\left(h=\frac{1}{2},Q=\pm1;\tau,z\right)\\ &\equiv \chi\left(p=\frac{i}{2\sqrt{3}},j=\pm2;\tau,z\right)+\\ &\quad+\chi\left(p=\frac{i}{2\sqrt{3}},j=\mp4;\tau,z\right).\end{aligned} \tag{D.13}$$

Therefore, any state of the form $|B;(m_+,m_-,n)\rangle$, $-m_+-m_-+n=0$ is reduced to a (linear combination of) class 2 state. Moreover, we can identify the minimal "basis" of the class 3 Cardy states that cannot be decomposed into other Cardy states as

$$|B;(1,0,n)\rangle\,,\qquad (n\geq2)\,,\qquad |B;(0,1,n)\rangle\,,\qquad (n\geq2)\,, \tag{D.14}$$

for $\mathbf{S}_+$, and

$$\begin{array}{llll}|B;(m_+,1,0)\rangle\,, & (m_+\geq1)\,, & |B;(1,m_-,0)\rangle\,, & (m_-\geq2)\,,\\ |B;(m_+,0,1)\rangle\,, & (m_+\geq2)\,, & |B;(0,m_-,1)\rangle\,, & (m_-\geq2)\,,\end{array} \tag{D.15}$$

for $\mathbf{S}_-$.

4. $\hat{c}=5$**:** We finally consider the special case $\hat{c}=5$ in which we have no transverse degrees of freedom. This case corresponds to $N=1$, $K=2$ ($\mathcal{Q}=2$) and quite reminiscent of the $\hat{c}=3$ case.

The class 1 and 2 states are written as

$$\begin{aligned}&\textbf{class 1 states:}\quad |B;r=0\rangle\equiv|B;O\rangle\,,\\ &\textbf{class 2 states:}\quad |B;p,j\rangle\\ &\qquad\left(j\in\mathbb{Z}_4\,,\quad \frac{p^2}{2}>-\frac{1}{8}\quad (j=0,1,-1)\,,\quad \frac{p^2}{2}\geq0\quad (j=2)\right).\end{aligned} \tag{D.16}$$

The candidates of class 3 states are

$$[(0,2),\,(0,3)]\,,\qquad [(0,1),\,(0,4)]\,. \tag{D.17}$$

The first one is again reduced to a class 2 state because of the character identity

$$\chi_{\mathbf{M}}(0,2;\tau,z)+\chi_{\mathbf{M}}(0,3;\tau,z)=\chi(p=0,j=2;\tau,z)(\equiv \mathrm{Ch}^{(\mathrm{NS})}(h=1,|Q|=2;\tau,z)). \tag{D.18}$$

The self-overlap of the second one is calculated as

$$\begin{aligned}&e^{\pi\frac{5z^2}{T}}\,\langle B;(0,1),\,(0,4)|e^{-\pi TH^{(c)}}e^{i\pi z(J_0+\tilde{J}_0)}|B;(0,1),\,(0,4)\rangle=\\ &\qquad=\int_0^\infty dp\sum_{j=0,\pm1,2}\rho_j(p)\chi(p,j;it,z')\,,\\ &\rho_0(p)=2\delta(p)+4\int_0^\infty dp'\,\big(\coth(\pi p')\coth(2\pi p')-1\big)\cos(2\pi pp')\,,\\ &\rho_1(p)=\rho_{-1}(p)=2\int_0^\infty dp'\,\frac{\cos(2\pi pp')}{\sinh(\pi p')\sinh(2\pi p')}\\ &\rho_2(p)=2\delta(p)\,.\end{aligned} \tag{D.19}$$

The spectral densities are indeed positive, and hence $|B;(0,1),\,(0,4)\rangle$ are the Cardy state.

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